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## An example of an immersed complete genus one minimal surface in $\mathbb{R}^3$ with two convex ends

Barbara Nelli

**Abstract.** We prove the existence of a compact genus one immersed minimal surface  $M$ , whose boundary is the union of two immersed locally convex curves lying in parallel planes.  $M$  is a part of a complete minimal surface with two finite total curvature ends.

**Mathematics Subject Classification (1991).** 53A10, 53C42.

**Keywords.** Minimal surface, convex boundary, Weierstrass representation, elliptic functions.

### 1. Introduction

In 1978 Meeks conjectured that a connected minimal surface bounded by two convex curves in two parallel planes is topologically an annulus; hence it has genus zero. The conjecture has never been proved and the most general result, due to Schoen, is the following.

Let  $\Gamma = \Gamma_1 \cup \Gamma_2$  be any boundary consisting of two Jordan curves in parallel planes; assume that  $\Gamma$  is invariant by reflection through two planes  $P_1, P_2$  orthogonal to the planes of the  $\Gamma_i$  and that both  $P_1$  and  $P_2$  divide  $\Gamma$  into pieces which are graphs with locally bounded slope over the dividing plane. Then any minimal surface spanning  $\Gamma$  is topologically an annulus and is an embedded surface meeting each parallel plane between the planes of the  $\Gamma_i$  in smooth Jordan curves.

In particular, if  $\Gamma_1$  and  $\Gamma_2$  are circles such that the line joining their centers is perpendicular to the planes in which they lie, then  $M$  is a catenoid (cf. [Sc]).

In 1991, Meeks and White studied the space of minimal annuli bounded by convex curves in parallel planes (cf. [MW]).

In this paper we prove the existence of a compact genus one immersed minimal surface  $M$ , whose boundary is the union of two immersed locally convex curves lying in parallel planes. In fact  $M$  is a part of a complete minimal surface with two finite total curvature ends.

The method we use to construct our surface is the following.

It is well known that a minimal surface of genus  $g$  and  $k$  ends can be described

by its Weierstrass representation, that is a triple  $\{\overline{R} \setminus [p_1, \dots, p_k], \omega = f dz, g\}$ , where  $\overline{R}$  is a compact Riemann surface of genus  $g$ ,  $p_1, \dots, p_k$  are points in  $\overline{R}$ ,  $\omega$  is a holomorphic differential on  $R$  and  $g$  is a meromorphic function on  $R$ .

In our setting  $\overline{R}$  is a torus, so we can choose  $f$  and  $g$  to be elliptic functions. For references about the use of elliptic functions in the Weierstrass representation, see [A], [A1], [C], [C1], [R]).

I would like to thank Professor Harold Rosenberg for his continual encouragement and advice.

## 2. Statement of results

Consider the lattice  $L(1, i)$  on  $\mathbb{C}$  generated by 1 and  $i$  and let  $T^2$  be the torus  $\mathbb{C}/L(1, i)$ . Let  $\pi : \mathbb{C} \rightarrow T^2$  be the standard projection to the quotient and set  $p_o = \pi(0)$ ,  $p_1 = \pi(\frac{1}{2})$ ,  $p_2 = \pi(\frac{1+i}{2})$  and  $p_3 = \pi(\frac{i}{2})$ . Finally, let  $\wp$  be the Weierstrass function associated to the lattice  $L(1, i)$  and  $\wp'$  its derivative.

**Theorem 2.1.** *Let  $f, g : T^2 \setminus \{p_o, p_2\} \rightarrow \mathbb{C}$  be the two meromorphic functions defined by*

$$f = \wp^2 \quad g = \frac{\alpha \wp'}{\wp^3}$$

where  $\alpha$  is a real constant depending only on  $L(1, i)$  and  $\wp$ .

*Then  $\{T^2 \setminus [p_o, p_2], f dz, g\}$  is the Weierstrass representation of a complete genus one immersed minimal surface  $M$  with finite total curvature.*

**Remark 2.2.** The ends of  $M$  cannot be embedded. In fact, if a complete finite total curvature minimal surface has two embedded ends, it is a catenoid (cf. [Sc]).

The functions  $f$  and  $g$  extend meromorphically to  $T^2$  and we have  $g(p_o) = 0$  and  $g(p_2) = \infty$ . Hence the limit normal vector at both ends of  $M$  is vertical. Then we have the following result.

**Theorem 2.3.** *There exists a positive constant  $c \in \mathbb{R}$  such that  $M \cap \{|x_3| \leq c\}$  is a compact genus one immersed minimal surface having the property that each of the boundary curves  $M \cap \{x_3 = \pm c\}$  is a compact locally convex immersed curve.*

## 3. Proof of the theorems

We list some useful classical properties of the function  $\wp$  (cf. [B], [WW]).

By abuse of notation, we often identify points of  $\mathbb{C}$  with points of  $T^2$ . Let  $'$  be the differentiation with respect to the variable  $z \in \mathbb{C}$ .

(i)  $\wp$  is even and  $\wp'$  is odd. We have  $\wp(z), \wp'(z) \in \mathbb{R}$  when  $z \in \mathbb{R}$ ,  $\wp(p_1) = e_1 \in \mathbb{R}_+^*$ ,  $\wp(p_2) = 0$  and  $\wp(p_3) = -e_1$ .

The following identities hold:

(ii)  $(\wp')^2 = 4\wp(\wp^2 - e_1^2)$ ,  $\wp'' = 2(3\wp^2 - e_1^2)$ .

(iii)  $\wp(z + p_1) = \frac{e_1(\wp(z) + e_1)}{\wp(z) - e_1}$ ,  $\wp(z + p_3) = \frac{e_1(\wp(z) - e_1)}{\wp(z) + e_1}$ ,  $\wp(z + p_2) = -\frac{e_1^2}{\wp(z)}$ .

(iv)  $\wp'(z + p_2) = e_1^2 \frac{\wp'(z)}{\wp(z)^2}$ .

(v)  $\wp(iz) = -\wp(z)$ ,  $\wp'(iz) = i\wp'(z)$ .

(vi) The local expansion of  $\wp$  and  $\wp'$  around  $p_o$  is

$$\wp(z) = \frac{1}{z^2} + \frac{e_1^2}{5}z^2 + O(z^6),$$

$$\wp'(z) = -\frac{2}{z^3} + \frac{2e_1^2}{5}z + O(z^5).$$

*Proof of Theorem 2.1.* It is sufficient to prove that the following conditions are satisfied.

(A)  $z$  is a pole of order  $m$  of  $g \iff z$  is a zero of order  $2m$  of  $f$ .

(B)  $\int_\gamma (1 + |g|^2)|f| = \infty$  for every divergent path  $\gamma$  in  $M$ .

(C)  $\operatorname{Re} \int_\gamma fg = 0$  and  $\int_\gamma fg^2 = \overline{\int_\gamma f}$  for every closed path in  $M$ .

Zeros and poles of  $f, g, fg, fg^2$  in a fundamental region are as in figure 1.

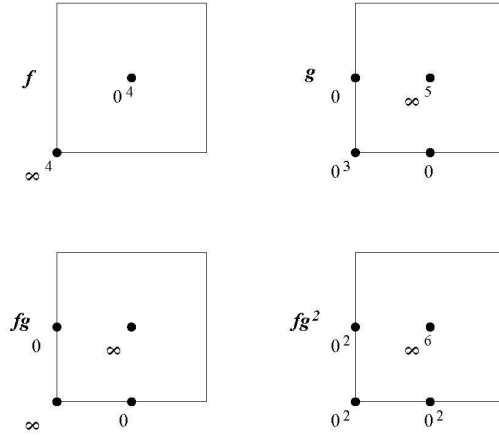


Figure 1.

The function  $g$  does not have poles in  $T^2 \setminus \{p_o, p_2\}$ , hence condition (A) is satisfied.

The expression of the metric on  $M$  in terms of  $\wp$  is

$$ds = \left(1 + \alpha^2 \frac{|\wp'|^2}{|\wp|^6}\right) |\wp|^2$$

hence the metric is complete at the ends and condition (B) is satisfied.

We must verify (C) on paths that are not homologous to 0 in  $T^2 \setminus \{p_o, p_2\}$ , i.e. paths around  $p_o$  and  $p_2$  and paths that generate the homology of  $T^2$ . Denote by  $\alpha(p_o)$  and  $\alpha(p_2)$  any closed path around  $p_o$  and  $p_2$  respectively, and by  $\gamma_1$  and  $\gamma_2$  the following paths generating the homology of  $T^2$ :

$$\gamma_1(t) = \frac{i}{4} + t \quad t \in [0, 1]$$

$$\gamma_2(t) = \frac{1}{4} + it \quad t \in [0, 1]$$

The functions  $f$  and  $fg^2$  are even, so they have no residue at  $p_o$ , i.e.

$$\int_{\alpha(p_o)} fg^2 = \int_{\alpha(p_o)} f = 0$$

Furthermore

$$\operatorname{Re} \int_{\alpha(p_o)} fg = \operatorname{Re} \int_{\alpha(p_o)} \frac{\alpha \wp'}{\wp} = \operatorname{Re} \left[ \operatorname{Res}_{p_o} \left( 2\pi i \alpha \frac{\wp'}{\wp} \right) \right]$$

By the local expansion of  $\wp$  and  $\wp'$  around 0 we have that  $\operatorname{Res}_{p_o} (2\pi i \alpha \frac{\wp'}{\wp}) = -4\pi i \alpha$ , hence for  $\alpha \in \mathbb{R}$  we have

$$\operatorname{Re} \int_{\alpha(p_o)} fg = 0$$

By (iii) and (iv) we have

$$f(z + p_2) = \frac{e_1^4}{\wp^2(z)},$$

$$fg^2(z + p_2) = \frac{\alpha^2}{e_1^4} (\wp'(z))^2.$$

Hence  $f(z + p_2)$  and  $fg^2(z + p_2)$  are even functions of  $z$  and this gives

$$\int_{\alpha(p_2)} fg^2 = \int_{\alpha(p_2)} f = 0.$$

By (iii) and (iv) we have

$$fg(z + p_2) = -\alpha \frac{\wp'(z)}{\wp(z)}.$$

Hence, by the computation above, for  $\alpha \in \mathbb{R}$  we have

$$\operatorname{Re} \int_{\alpha(p_2)} fg = 0.$$

Now we verify (C) over  $\gamma_1$  and  $\gamma_2$ . We have

$$\operatorname{Re} \int_{\gamma_i} fg = \operatorname{Re} \int_{\gamma_i} \alpha \frac{\wp'}{\wp} = \alpha [\ln |\wp|]_{\gamma_i(0)}^{\gamma_i(1)} = 0$$

by periodicity of  $\wp$ , as  $\alpha$  is real.

Integral of  $f$  over  $\gamma_1$  : by Cauchy theorem and periodicity we can move  $\gamma_1$  up to the segment from  $p_3$  to  $p_3 + 1$ , hence

$$\int_{\gamma_1} f = \int_0^1 f(p_3 + t) dt = \int_0^1 e_1^2 \frac{(\wp(t) - e_1)^2}{(\wp(t) + e_1)^2} dt$$

where the last equality is given by (iii).

Integral of  $f$  over  $\gamma_2$  : we can move  $\gamma_2$  to the vertical segment from  $p_1$  to  $p_1 + i$ , hence by (iii) and (iv)

$$\int_{\gamma_2} f = \int_0^1 f(p_1 + t) i dt = i \int_0^1 e_1^2 \frac{(\wp(t) - e_1)^2}{(\wp(t) + e_1)^2} dt.$$

Integral of  $fg^2$  over  $\gamma_1$  : we can move  $\gamma_1$  down to the real segment from  $p_o$  to  $p_o + 1$ , hence

$$\int_{\gamma_1} fg^2 = \int_0^1 f(t) g^2(t) dt = \int_0^1 \alpha^2 \frac{\wp'(t)^2}{\wp(t)^4} dt.$$

Integral of  $fg^2$  over  $\gamma_2$  : we can move  $\gamma_2$  to the vertical segment from  $p_o$  to  $p_o + i$ , hence

$$\int_{\gamma_2} fg^2 = \int_0^1 f(it) g^2(it) i dt = -i \int_0^1 \alpha^2 \frac{\wp'(t)^2}{\wp(t)^4} dt.$$

Then  $\alpha$  must satisfy

$$\alpha^2 \int_0^1 \frac{\wp'(t)^2}{\wp(t)^4} dt = \int_0^1 e_1^2 \frac{(\wp(t) - e_1)^2}{(\wp(t) + e_1)^2} dt.$$

If  $t \in \mathbb{R}$  we have  $\varphi(t), \varphi'(t) \in \mathbb{R}$ , hence the two integrals involved in the definition of  $\alpha$  are positive real numbers. Furthermore they are convergent, so  $\alpha \in \mathbb{R}$ .

Since  $g$  and  $f$  extend meromorphically to  $T^2$ ,  $M$  has finite total curvature.  $\square$

Before proving Theorem 2.3 we need the following lemma.

**Lemma 3.1.** *Consider a minimal surface  $M$  with Weierstrass representation given by  $\{f dz, g\}$  such that the vector corresponding to  $g(0)$  is parallel to the  $x_3$ -axis. Then the planar curvature of the intersection curves of  $M$  with the horizontal planes is*

$$k = \frac{1}{|f^2 g|(1 + |g|^2)} \operatorname{Re} \left( \overline{f g} \frac{g'}{g} \right).$$

*Proof.* Let  $\theta = \arg g$  and  $s$  be the arc length of the curve  $M \cap \{x_3 = c\}$ ; then  $k(s) = \frac{d\theta}{ds}$ . As  $\arg g = \operatorname{Im}(\ln g)$ , we have

$$k(s) = \frac{d \operatorname{Im} \ln g}{ds} = \operatorname{Im} \left( \frac{d \ln g}{dz} \frac{dz}{ds} \right) = \operatorname{Im} \left( \frac{g'}{g} \frac{dz}{ds} \right).$$

By the Weierstrass representation we have

$$x_3 = \operatorname{Re} \int f g.$$

Hence, on the curve  $M \cap \{x_3 = c\}$ ,  $\frac{dz}{ds}$  must satisfy

$$0 = \frac{d}{ds} \operatorname{Re} \int f g = \frac{1}{2} \operatorname{Re} \left( f g \frac{dz}{ds} \right).$$

By a straightforward computation we obtain

$$\frac{dz}{ds} = \frac{i}{(1 + |g|^2)|f|} \frac{\overline{f g}}{|f g|}.$$

Then

$$k = \operatorname{Im} \left( \frac{i}{(1 + |g|^2)|f|} \frac{\overline{f g}}{|f g|} \frac{g'}{g} \right) = \frac{1}{|f^2 g|(1 + |g|^2)} \operatorname{Re} \left( \overline{f g} \frac{g'}{g} \right).$$

$\square$

*Proof of Theorem 2.3.* The third coordinate of  $M$  is given by

$$x_3 = \operatorname{Re} \int f g = \operatorname{Re} \int \alpha \frac{\varphi'}{\varphi} = \alpha \ln |\varphi|,$$

since  $\alpha$  is real. Then, any level curve is given by  $|\wp| = c$  and next to the ends this is a compact immersed curve with only one component.

By a straightforward computation, we obtain

$$g'(z) = 2\alpha \left[ \frac{5e_1^2 - 3\wp(z)^2}{\wp(z)^3} \right],$$

$$\frac{g'(z)}{g(z)} = \frac{2(5e_1^2 - 3\wp(z)^2)}{\wp'(z)},$$

$$\overline{f(z)g(z)} = \overline{\alpha} \frac{\overline{\wp'(z)}}{\wp(z)}.$$

By using the expansion of  $\wp$  and  $\wp'$  at  $p_o$  we have

$$\overline{f(z)g(z)} \sim -2\frac{\overline{\alpha}}{\bar{z}},$$

$$\frac{g'(z)}{g(z)} \sim \frac{3}{z},$$

where  $\sim$  denotes equality between the principal parts of the functions in a neighborhood of zero. Hence the sign of the curvature of the level curve next to the end  $p_o$  is the same as the sign of

$$\operatorname{Re}\left(\frac{-6\overline{\alpha}}{\bar{z}z}\right) = -\frac{6\alpha}{|z|^2},$$

$\alpha$  being real.

We use the equality

$$\overline{f(z+p_2)g(z+p_2)} = -\overline{f(z)g(z)}$$

and the fact that in a neighborhood of zero we have

$$\frac{g'(z+p_2)}{g(z+p_2)} = \frac{2(5\wp(z)^2 - 3e_1^2)}{\wp'(z)} \sim -\frac{5}{z},$$

to conclude that the sign of the curvature of the level curve next to the end  $p_2$  is the same as the sign of

$$\operatorname{Re}\left(\frac{-10\overline{\alpha}}{\bar{z}z}\right) = -\frac{10\alpha}{|z|^2}$$

since  $\alpha$  is real.

Thus, if we choose a negative  $\alpha$ , the level curves are locally convex next to the two ends of  $M$ .  $\square$



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