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Deforming abelian SU(2)–representations of knot groups

Michael Heusener and Jochen Kroll*

Abstract. The aim of this paper is to generalize a theorem of C. D. Frohman and E. P. Klassen ([FK91]) concerning deformations of abelian SU(2)-representations of knot groups into non-abelian representations. The proof of our main theorem makes use of a generalization of a result of X.-S. Lin ([Lin92]) which should be interesting in itself.

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1. Introduction

The aim of this paper is to study the following question: when is an abelian representation of a knot group in SU(2) a limit point of non-abelian representations?

Let $k \subset S^3$ be a tame knot and let $G := \pi_1(S^3 \setminus k)$ be its group. A homomorphism $\rho\colon G \to \mathrm{SU}(2)$ is called abelian if and only if its image is abelian. The space of abelian conjugacy classes of representations is parameterized by the closed interval $[0, \pi]$. More precisely, let $m \in G$ be a meridian and let $\alpha \in [0, \pi]$ be given. We define an abelian representation $\rho_\alpha\colon G \to \mathrm{SU}(2)$ by $\rho_\alpha(m) = \mathbf{e}^{\mathbf{i}\alpha}$, where

$$\mathbf{e}^{i\alpha} := \begin{pmatrix} e^{\mathbf{i}\alpha} & 0 \\ 0 & e^{-\mathbf{i}\alpha} \end{pmatrix}.$$

We denote the Alexander polynomial of k by Δ_k. If ρ_α is a limit of non-abelian representations then $\Delta_k(e^{2\mathbf{i}\alpha}) = 0$ (see theorem 2.1). It is conjectured that this condition is also sufficient.

C. Frohman and E. Klassen (see [FK91]) proved the conjecture under the assumption that $e^{2\mathbf{i}\alpha}$ is a simple root of Δ_k.

The aim of this paper is to prove the following theorem:

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Theorem 1.1. *Let $k \subset S^3$ be a knot and let $\alpha \in [0, \pi]$ be such that $\Delta_k(e^{2\mathbf{i}\alpha}) = 0$. If the signature function $\sigma_k \colon S^1 \longrightarrow \mathbb{Z}$ changes its value at $e^{2\mathbf{i}\alpha}$ then the abelian representation ρ_α is an endpoint of an arc of non-abelian representations $G \longrightarrow$* SU(2).

The signature function changes its value at $\omega \in S^1$ if ω is a root of Δ_k of odd multiplicity (see section 2). Therefore, theorem 1.1 generalizes theorem 1.1 in [FK91]. However, there are non-abelian deformations of abelian representations which are not detected by theorem 1.1. The 2-bridge knot $\mathfrak{b}(49, 17)$ has an abelian representation which is the limit of non-abelian representations but the signature function does not change its value at the corresponding zero of the Alexander polynomial (see [Bur90]).

The proof of theorem 1.1 makes use of a generalization of a result of X.-S. Lin (see [Lin92]): let G be a knot group and let $m \in G$ be a meridian. A representation $\rho \colon G \longrightarrow \mathrm{SU}(2)$ is called trace-free if $\mathrm{tr}\,\rho(m) = 0$. In [Lin92] Lin defined an intersection number of the representation spaces corresponding to a braid representative of the knot. This number turns out to be a knot invariant denoted by $h(k)$. Roughly speaking, $h(k)$ is the number of conjugacy classes of non-abelian trace-free representations $G \longrightarrow \mathrm{SU}(2)$ counted with sign. Moreover, Lin established the relation

$$h(k) = \frac{1}{2}\sigma(k).$$

It was remarked by D. Ruberman that the construction can be generalized to representations of knot groups with the trace of the meridians fixed. In section 4 we shall establish this generalization. More precisely, for a given $\alpha \in (0, \pi)$, we will define an integer invariant $h^\alpha(k)$. This invariant counts the conjugacy classes of non-abelian representations $G \longrightarrow \mathrm{SU}(2)$, such that $\mathrm{tr}\,\rho(m) = 2\cos\alpha$ (note that $h(k) = h^{\pi/2}(k)$). Since the definition of $h^\alpha(k)$ is straightforward we will only explain the set-up. However, the significant part is the computation of $h^\alpha(k)$ for $\alpha \neq \pi/2$. We shall present all the details needed to prove the following theorem.

Theorem 1.2. *Let k be a knot such that $\Delta_k(e^{2\mathbf{i}\alpha}) \neq 0$. Then*

$$h^\alpha(k) = \frac{1}{2}\sigma_k(e^{2\mathbf{i}\alpha}).$$

The proof of theorem 1.1 together with theorem 1.2 implies the following:

Corollary 1.3. *Let $k \subset S^3$ be a knot. If there is an $\alpha \in [0, \pi]$ such that $\Delta_k(e^{2\mathbf{i}\alpha}), \sigma_k(e^{2\mathbf{i}\alpha}) \neq 0$ then there exists an arc of irreducible* SU(2) *representations $G \longrightarrow$* SU(2).

This paper is organized as follows: In section 2 the basic notation and facts are presented. The proof of theorem 1.1 and the definition of $h^\alpha(k)$ are contained in section 3. The last section includes the computation of the invariant $h^\alpha(k)$.

Remark 1.4. Theorem 1.1 was proved independently by C. Herald using gauge theory (see [Her97]).

2. Notations and facts

In this section we present the notation and facts which are needed in the sequel.

2.1. The signature function

Let $k \subset S^3$ be a tame knot and let F be a Seifert surface for k. Since F is oriented we have a normal direction and we can push cycles on F along this normal direction into the complement of F. This defines a homomorphism $H_1(F) \to H_1(S^3 \smallsetminus F)$, $x \mapsto x^+$ (here $H_*(X) := H_*(X, \mathbb{Z})$). The Seifert pairing $H_1(F) \otimes H_1(F) \to \mathbb{Z}$ is given by $x \otimes y \mapsto lk(x, y^+)$ where lk is the linking number in S^3. By fixing a basis $\{a_i \mid 1 \leq i \leq 2g\}$ of $H_1(F)$ the pairing is described by a $2g \times 2g$ matrix V over $\mathbb{Z}$ (g denotes the genus of F). We call V a Seifert matrix for k. The antisymmetric matrix $V - V^T$ (V^T is the transposed matrix of V) is the intersection matrix of the basis $\{a_i\}$ in $H_1(F)$ (see [BZ85] for details).

The normalized Alexander polynomial for k is given by $\Delta_k := \det(t^{1/2}V - t^{-1/2}V^T)$. Here, normalized stands for $\Delta_k(t) = \Delta_k(t^{-1})$ and $\Delta_k(1) = 1$.

Let $\omega \neq 1$ be a complex number. We now consider the hermitian matrix $H(\omega) := (1-\omega)V + (1-\bar{\omega})V^T$. The ω-signature $\sigma_k(\omega)$ of k is defined to be the signature of $H(\omega)$ i.e $\sigma_k(\omega) = \operatorname{sig}(H(\omega))$. If $\omega \in S^1 \smallsetminus \{1\}$ we have

$$\begin{aligned} H(\omega) &= (1-\omega)V + (1-\bar{\omega})V^T \\ &= (\omega^{-1/2} - \omega^{1/2})(\omega^{1/2}V - \omega^{-1/2}V^T). \end{aligned} \tag{1}$$

The Levine–Tristram signature function is the map $\sigma_k\colon S^1 \to \mathbb{Z}$ given by $\sigma_k\colon \omega \mapsto \sigma_k(\omega)$ if $\omega \neq 1$ and $\sigma_k\colon 1 \mapsto 0$. Let $Z_k := \{\omega \in S^1 \mid \Delta_k(\omega) = 0\}$ be the set of zeros of Δ_k on the unit circle. It follows from equation (1) that the signature function is constant on the components of $S^1 \smallsetminus Z_k$. For a given $e^{2\mathbf{i}\beta} \in Z_k$ we use the expression "*σ_k change its value at $e^{2\mathbf{i}\beta}$*" or "*σ_k jumps at $e^{2\mathbf{i}\beta}$*" if

$$\lim_{t\downarrow\beta} \sigma_k(e^{2\mathbf{i}t}) \neq \lim_{t\uparrow\beta} \sigma_k(e^{2\mathbf{i}t}).$$

Moreover, it can be seen that $\sigma_k(\omega) = 0$ if ω lies in a small neighborhood of 1 (for details see [Gor77] and [Kau87]).

2.2. Representation spaces

Let G be a finitely generated group. The space of all representations of G in $\mathrm{SU}(2)$ is denoted by $R(G) := \operatorname{Hom}(G, \mathrm{SU}(2))$. Note that $R(G)$ is a topological space via the compact open topology where G carries the discrete and SU(2) the usual topology. A representation $\rho \in R(G)$ is called abelian (resp. central), (resp. trivial) if and only if its image is an abelian (resp. central), (resp. trivial) subgroup of SU(2). Note that $\rho \in R(G)$ is abelian if and only if it is reducible. The set of abelian representations is denoted by $S(G)$ and the set of central representations by $C(G)$. Two representations $\rho, \varrho \in R(G)$ are said to be conjugate ($\rho \sim \varrho$) if and only if they differ by an inner automorphism of SU(2). The group $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm 1\}$ acts on $R(G)$ via conjugation. Two representations are in the same SO(3)–orbit if and only if they are conjugate. Let $\widetilde{R}(G) := R(G) \setminus S(G)$ be the set of non-abelian representations. The space of (non-abelian) conjugacy classes of representations from G into SU(2) is denoted by $\mathfrak{R}(G)$ ($\widehat{R}(G)$) i.e.

$$\mathfrak{R}(G) := R(G)/\mathrm{SO}(3) \quad \text{and} \quad \widehat{R}(G) := \widetilde{R}(G)/\mathrm{SO}(3).$$

We can think of the map $\widetilde{R}(G) \to \widehat{R}(G)$ as a principal SO(3)–bundle (see [GM92, 3.A] for details). The spaces $\widehat{R}(G)$ and $\mathfrak{R}(G)$ are semi-algebraic sets. Here a subset of $\mathbb{R}^n$ is called *semi-algebraic* if it is a finite union of finite intersections of sets defined by a polynomial equation or inequality (see [Heu97] for details).

If $k \subset S^3$ is a knot then let $R(k)$ be short for $R(\pi_1(S^3 \setminus k))$. As we need the following theorem several times in the sequel we state it here.

Theorem 2.1. *Let $k \subset S^3$ be a knot and let $\rho_\alpha \in S(k)$ be given. If $\Delta_k(e^{2\mathbf{i}\alpha}) \neq 0$, then a sufficiently small neighborhood of ρ_α consists entirely of points of $S(k)$.*

Proof. See [Kla91, Theorem 19]. □

2.3. Quaternions

During this paper it is sometimes more convenient to work with the quaternions (which we denote by $\mathbb{H}$). We identify SU(2) with the unit quaternions $\mathrm{Sp}(1) \subset \mathbb{H}$, the isomorphism is given by

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mapsto a + b\mathbf{j}.$$

The Lie algebra of Sp(1) is the set $\mathbb{E}$ of pure quaternions and Sp(1) acts via Ad on $\mathbb{E}$ i.e. $\mathrm{Ad}(q)X = qXq^{-1}$ for $q \in \mathrm{Sp}(1)$ and $X \in \mathbb{E}$. The intersection $\mathbb{E} \cap \mathrm{Sp}(1)$ – the set of pure unit quaternions – which is homeomorphic to the 2–sphere will be denoted by S^2. More general we consider the argument function arg: $\mathrm{SU}(2) \twoheadrightarrow [0, \pi]$ given by $\arg(A) = \arccos(\mathrm{tr}(A)/2)$. For $\alpha \in (0, \pi)$ we have $\Sigma_\alpha := \arg^{-1}(\alpha)$ is a 2-sphere and $S^2 = \Sigma_{\pi/2}$.

Given two elements $X, Y \in \mathbb{E}$, there is a product formula: $X \cdot Y = -\langle X, Y\rangle + X \times Y$ where $\langle X, Y\rangle$ denotes the scalar product of X and Y and $X \times Y$ their vector product in $\mathbb{E}$. Note that $\mathrm{Ad}(q)$ preserves the scalar product. For $\mathbf{e}^{\mathbf{i}\alpha} \in \mathrm{SU}(2)$ we identify the tangent space $T_{\mathbf{e}^{\mathbf{i}\alpha}}(\Sigma_\alpha) = \mathrm{span}(\mathbf{j}, \mathbf{k})$ with $\mathbb{C}$ via the multiplication by $-\mathbf{j}$. Under this identification the action of $\mathrm{Ad}(\mathbf{e}^{\mathbf{i}\alpha})$ transforms into rotation about the angle 2α i.e multiplication by $e^{2\mathbf{i}\alpha}$.

For each quaternion $q \in \mathrm{Sp}(1)$ there is an angle α, $0 \leq \alpha \leq \pi$, and $Q \in S^2$ such that $q = \cos\alpha + \sin\alpha\, Q$. The pair (α, Q) is unique if and only if $q \neq \pm 1$. Let $(\alpha, Q) := \cos\alpha + \sin\alpha\, Q$ for short.

2.4. Burau-matrix

Let $\mathfrak{B}_n$ be the braid group of rank n with the standard generators $\sigma_1, \dots, \sigma_{n-1}$. For a given $\sigma \in \mathfrak{B}_n$ we denote by $\Phi^\sigma := (\Phi^\sigma_{i,j})_{1 \leq i,j \leq n} \in \mathrm{GL}(n, \mathbb{Z}(t))$ its Burau-matrix and by $\phi^\sigma \in \mathrm{GL}(n-1, \mathbb{Z}(t))$ its reduced Burau-matrix (see [BZ85]). We write Φ^σ in the form

$$\Phi^\sigma = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$$

where $\mathbf{D} = \mathbf{D}(t)$ is a $(n-2) \times (n-2)$ matrix. The matrix $C^\sigma(t) := (\Phi^\sigma - \mathbf{E})$ is a Jacobian for the closed braid $\sigma^\wedge$ where $\mathbf{E}$ denotes the identity matrix (see [BZ85]). Let $c^\sigma_{i,j}(t)$ be the determinant of the matrix which is obtained from $C^\sigma(t)$ by omitting its i-th row and its j-th column. It is convenient to state the following lemma which will be used in the sequel.

Lemma 2.2. *Let the $c^\sigma_{i,j} := c^\sigma_{i,j}(t)$ be defined as above. Then*

1. $c^\sigma_{l,m} = (-1)^{m+m'+l+l'} t^{l-l'} c^\sigma_{l',m'}$
2. $\det(\mathbf{E} - \phi^\sigma(t)) = \frac{t^n - 1}{t-1} c^\sigma_{1,n}$
3. $c^{\sigma_1^2\sigma}_{1,1} = t^2 c^\sigma_{1,1} + (t-1)\det(\mathbf{D}(t) - \mathbf{E})$.

Proof. The lemma is proved using the identities

$$\sum_{j=1}^n \Phi^\sigma_{i,j} = 1 \text{ and } \sum_{i=1}^n t^{i-1}\Phi^\sigma_{i,j} = t^{j-1}$$

(see [BZ85] for details). □

3. Proof of the main Theorem

Let $\sigma \in \mathfrak{B}_n$ be given and denote by $\sigma^\wedge$ the closed n-braid defined by σ. Let F_n be a free group with basis $\mathrm{S} = \{s_1, \dots, s_n\}$. The braid σ induces a braid automorphism (still denoted by σ) $\sigma\colon F_n \to F_n$. It follows that σ induces a diffeomorphism (still denoted by σ) of $\mathrm{SU}(2)^n$ i.e.

$$\sigma(A_1, \dots, A_n) =: (\sigma(A_1), \dots, \sigma(A_n)).$$

Example 3.1. Let $\sigma = \sigma_1^{-2} \in \mathfrak{B}_2$. The $\sigma(A_1, A_2) =: (\sigma(A_1), \sigma(A_2))$ where $\sigma(A_1) = A_2^{-1} A_1 A_2$ and $\sigma(A_2) = A_2^{-1} A_1^{-1} A_2 A_1 A_2$.

Note that the equation $\prod_{i=1}^n A_i = \prod_{i=1}^n \sigma(A_i)$ always holds.

It was observed by Lin that the fixed point set of σ: $\mathrm{SU}(2)^n \to \mathrm{SU}(2)^n$ can be identified with $R(\sigma^\wedge)$ [Lin92, Lemma 1.2]. Let $(A_1, \ldots, A_n) \in \mathrm{Fix}(\sigma)$ be given. It follows that $\operatorname{tr} A_i = \operatorname{tr} A_j$ if $\sigma^\wedge$ is a knot. Therefore we are interested in the following space:

$$R_n := \{(A_1, \ldots, A_n) \in \mathrm{SU}(2)^n \mid \operatorname{tr}(A_i) = \operatorname{tr}(A_j),\ 1 \le i, j \le n\} \setminus \{\pm(\mathbf{E}, \ldots, \mathbf{E})\}.$$

Since $\sigma(R_n) = R_n$ we obtain a diffeomorphism σ: $R_n \to R_n$. Its fixed point set can be identified with $R(\sigma^\wedge) \setminus C(\sigma^\wedge)$.

Remark 3.2. A central representation of a knot group can never be a limit of non-abelian representations since $\Delta_k(1) = 1$ (see theorem 2.1).

For a given $\alpha \in (0, \pi)$ let

$$R_n^\alpha := \{(A_1, \ldots, A_n) \mid \operatorname{tr}(A_i) = 2\cos\alpha,\ 1 \le i \le n\} \subset R_n.$$

The set $R_n^\alpha \subset R_n$ is a submanifold of codimension one. Let us consider the following subspaces of R_{2n}:

$$\begin{aligned}
H_n &:= \{(A_1, \ldots, A_n, B_1, \ldots, B_n) \in R_{2n} | A_1 \cdots A_n = B_1 \cdots B_n\},\\
\Lambda_n &:= \{(A_1, \ldots, A_n, A_1, \ldots, A_n) \in R_{2n}\},\\
\Gamma_\sigma &:= \{(A_1, \ldots, A_n, \sigma(A_1), \ldots, \sigma(A_n)) \in R_{2n}\},\\
S_n &:= \{(A_1, \ldots, A_{2n}) \in R_{2n} \mid A_i A_j = A_j A_i,\ 1 \le i, j \le n\}.
\end{aligned}$$

Moreover, for $\Theta \in \{H_n, \Gamma_\sigma, \Lambda_n, S_n\}$ let $\Theta^\alpha := \Theta \cap (R_n^\alpha \times R_n^\alpha)$. It is obvious that the fixed point set of $\sigma : R_n \to R_n$ is given by $\Gamma_\sigma \cap \Lambda_n \subset H_n$.

The sets $S_n \subset R_{2n}$ and $S_n^\alpha \subset R_n^\alpha \times R_n^\alpha$ are the subsets corresponding to the abelian representations. Each element of S_n^α is conjugate to an element of the form $(\mathbf{e}^{\epsilon_1 \mathbf{i}\alpha}, \ldots, \mathbf{e}^{\epsilon_{2n} \mathbf{i}\alpha})$ where $\epsilon_i \in \{\pm 1\}$.

Lemma 3.3. *Let $n \in \mathbb{Z}$, $n \ge 2$ be given. The set $H_n \setminus S_n$ is a smooth manifold of dimension $4n-2$ and $H_n^\alpha \setminus S_n^\alpha$ is a smooth manifold of dimension $4n-3$. Moreover, $H_n^\alpha \subset H_n$ is a submanifold of codimension one.*

Proof. Let f_n: $R_{2n} \to \mathrm{SU}(2)$ be given by

$$f_n\colon (A_1, \ldots, A_n, B_1, \ldots, B_n) \mapsto A_1 \cdots A_n B_n^{-1} \cdots B_1^{-1}$$

and denote by f_n^α its restriction to $R_n^\alpha \times R_n^\alpha$. Note that $R_n^\alpha \times R_n^\alpha \subset R_{2n}$ is a submanifold of codimension one.

For a given point $(\mathbf{A},\mathbf{B}) \in H_n^\alpha \setminus S_n^\alpha$ the derivative $D_{(\mathbf{A},\mathbf{B})} f_n^\alpha$ is surjective. This can be seen by a direct calculation (see [Lin92] or [Heu97]). The conclusion of the lemma follows from this fact. □

By fixing an orientation of SU(2) we orient the 2-spheres $\Sigma_\alpha = \arg^{-1}(\alpha)$ in the following way: the normal bundle ν_α of $\Sigma_\alpha \subset \mathrm{SU}(2)$ is oriented by pulling back the orientation $-dx$ of $[0,\pi]$. We choose the orientation of Σ_α such that the orientations of the short exact sequence

$$0 \to T\Sigma_\alpha \to T_{\Sigma_\alpha}\mathrm{SU}(2) \to \nu_\alpha \to 0$$

fit together. The space $R_n^\alpha \cong \Sigma_\alpha^n$ is then oriented via the product orientation. Moreover, the orientation of $R_n \cong (0,\pi)\times(S^2)^n$ is obtained from the orientations of R_n^α and $[0,\pi]$. The manifolds $\Lambda_n, \Gamma_\sigma \cong R_n$ and $\Lambda_n^\alpha, \Gamma_\sigma^\alpha \cong R_n^\alpha$ are also oriented. By lemma 3.3 we can pull back the orientation of SU(2) to obtain an orientation of the normal bundle of $f_n^{-1}(\mathbf{E})\setminus S_n$ (resp. $(f_n^\alpha)^{-1}(\mathbf{E})\setminus S_n^\alpha$). This enables us to orient the manifolds $H_n \setminus S_n$ and $H_n^\alpha \setminus S_n^\alpha$. Now SO(3) acts fixed point free via conjugation on the oriented manifolds $\Theta \setminus S_n$ (resp. $\Theta^\alpha \setminus S_n^\alpha$) where $\Theta \in \{H_n, \Lambda_n, \Gamma_\sigma\}$. The action of SO(3) is orientation preserving (SO(3) is connected) and we obtain the following oriented manifolds:

$$\widehat{\Theta} := (\Theta \setminus S_n)/\mathrm{SO}(3) \text{ and } \widehat{\Theta}^\alpha := (\Theta^\alpha \setminus S_n^\alpha)/\mathrm{SO}(3).$$

Remark 3.4. Since $\dim R_{2n} = 4n+1$ and $\dim R_n^\alpha = 2n$ one has:

$$\dim \widehat{\Lambda}_n = 2n-2, \qquad \dim \widehat{\Gamma}_\sigma = 2n-2, \qquad \dim \widehat{H}_n = 4n-5$$

and

$$\dim \widehat{\Lambda}_n^\alpha = 2n-3, \qquad \dim \widehat{\Gamma}_\sigma^\alpha = 2n-3, \qquad \dim \widehat{H}_n^\alpha = 4n-6.$$

The intersection number

$$h^\alpha(\sigma) := \langle \widehat{\Lambda}_n^\alpha, \widehat{\Gamma}_\sigma^\alpha \rangle_{\widehat{H}_n^\alpha} \in \mathbb{Z}$$

is well defined if the intersection $\widehat{\Lambda}_n^\alpha \cap \widehat{\Gamma}_\sigma^\alpha \subset \widehat{H}_n^\alpha$ is compact. It can be shown that h^α is indeed a knot invariant.

Proposition 3.5. *Let $\sigma \in \mathfrak{B}_n$ and $\tau \in \mathfrak{B}_m$ be given such that $\sigma^\wedge \cong \tau^\wedge$ as knots. If $\widehat{\Lambda}_n^\alpha \cap \widehat{\Gamma}_\sigma^\alpha$ is compact then $\widehat{\Lambda}_m^\alpha \cap \widehat{\Gamma}_\tau^\alpha$ is also compact and the intersection numbers $h^\alpha(\sigma)$ and $h^\alpha(\tau)$ are equal.*

Proof. Analogous to Lin's proof (see [Lin92, Theorem 18]). □

The following lemma gives a criteria for the compactness of the intersection $\widehat{\Lambda}_n^\alpha \cap \widehat{\Gamma}_\sigma^\alpha$.

Lemma 3.6. *Let $\sigma \in \mathfrak{B}_n$ be a braid such that $k := \sigma^\wedge$ is a knot. If $\Delta_k(e^{2\mathbf{i}\alpha}) \neq 0$ then*

$$\widehat{\Lambda}_n^\alpha \cap \widehat{\Gamma}_\sigma^\alpha \subset \widehat{H}_n^\alpha$$

is compact.

Proof. The conclusion of the lemma follows direct from theorem 2.1. □

Remark 3.7. It is possible for $\widehat{\Lambda}_n^\alpha \cap \widehat{\Gamma}_\sigma^\alpha \subset \widehat{H}_n^\alpha$ to be compact but $\Delta_k(e^{2\mathbf{i}\alpha}) = 0$. An example is given by the class of 2-bridge knots.

The space of non-abelian equivalence classes of representations $\widehat{R}(\sigma^\wedge)$ can be identified with the intersection $\widehat{\Lambda}_n \cap \widehat{\Gamma}_\sigma \subset \widehat{H}_n$. The intersection $\widehat{\Lambda}_n \cap \widehat{\Gamma}_\sigma$ is in general not compact. However, if the abelian representation ρ_α is not a limit of non-abelian representations the intersection $\widehat{\Lambda}_n^\alpha \cap \widehat{\Gamma}_\sigma^\alpha \subset H_n^\alpha$ is compact. Moreover, there exists an $\epsilon > 0$ such that

$$(\widehat{\Lambda}_n \cap \widehat{\Gamma}_\sigma) \cap \widehat{H}_n^{[\alpha-\epsilon,\alpha+\epsilon]}$$

is compact where

$$\widehat{H}_n^{[\alpha_1,\alpha_2]} := \bigcup_{\alpha \in [\alpha_1,\alpha_2]} \widehat{H}_n^\alpha.$$

This follows from the fact that $\mathfrak{R}(k)$ is a compact, semi-algebraic set.

Proposition 3.8. *Let $\sigma \in \mathfrak{B}_n$ be given such that $\sigma^\wedge$ is a knot. Moreover, assume that ρ_α is not a limit of non-abelian representations.*

Then there is an $\epsilon > 0$ such that $h^\alpha(\sigma) = h^\beta(\sigma)$ for $|\alpha - \beta| < \epsilon$.

Proof. Choose an $\epsilon > 0$ such that

$$(\widehat{\Lambda}_n \cap \widehat{\Gamma}_\sigma) \cap \widehat{H}_n^{\alpha,\epsilon}$$

is compact where $\widehat{H}_n^{\alpha,\epsilon} := \widehat{H}_n^{[\alpha-\epsilon,\alpha+\epsilon]}$.

In general we have $\widehat{\Gamma}_\sigma \pitchfork \widehat{H}_n^\beta$ for all $\beta \in (0,\pi)$. Let $\widehat{\Gamma}_\sigma^\alpha \rightsquigarrow \widetilde{\Gamma}_\sigma^\alpha$ be an isotopy with compact support such that $\widetilde{\Gamma}_\sigma^\alpha \pitchfork \widehat{\Lambda}_n^\alpha$. Extend this deformation to an isotopy $\widehat{\Gamma}_\sigma \rightsquigarrow \widetilde{\Gamma}_\sigma$ with compact support such that $\widetilde{\Gamma}_\sigma \pitchfork_{\widehat{H}_n^{\alpha,\epsilon}} \widehat{\Lambda}_n$ and $\widetilde{\Gamma}_\sigma \pitchfork \widehat{H}_n^\beta$ for all $\beta \in (0,\pi)$. For a given $\beta \in (0,\pi)$ let $\widetilde{\Gamma}_\sigma^\beta := \widetilde{\Gamma}_\sigma \cap \widehat{H}_n^\beta$. Note that $\widetilde{\Gamma}_\sigma^\beta \subset \widehat{H}_n^\beta$ is a $(2n-3)$-dimensional manifold and $\widehat{\Lambda}_n^\beta \pitchfork \widetilde{\Gamma}_\sigma^\beta$ if $|\alpha - \beta|$ is sufficiently small.

Now $\widehat{\Lambda}_n \cap \widetilde{\Gamma}_\sigma$ is an oriented one dimensional manifold in a neighborhood of $\widehat{H}_n^\alpha$. If β is close to α then the one dimensional manifold $\widehat{\Lambda}_n \cap \widetilde{\Gamma}_\sigma$ yields a bordism in $\widehat{H}_n$ between $\widehat{\Lambda}_n^\beta \cap \widetilde{\Gamma}_\sigma^\beta \subset \widehat{H}_n^\beta$ and $\widehat{\Lambda}_n^\alpha \cap \widetilde{\Gamma}_\sigma^\alpha \subset \widehat{H}_n^\alpha$.

It follows that $h^\alpha(\sigma) = h^\beta(\sigma)$. □

Proof of theorem 1.1. Assume that ρ_α is not the limit of non-abelian representations. By proposition 3.8 and by theorem 1.2 we obtain that the signature function does not jump at $e^{2\mathbf{i}\alpha}$. This contradicts the hypothesis of the theorem 1.1. □

A further consequence is the following:

Corollary 3.9. *Let $k \subset S^3$ be a knot. If $\Delta_k(e^{2\mathbf{i}\alpha}) \neq 0$ and if $\sigma_k(e^{2\mathbf{i}\alpha}) \neq 0$ then there is a non-abelian representation $\rho_0 \in \widehat{R}(k)$ such that* $\operatorname{tr}\rho_0(m) = 2\cos\alpha$. *Moreover, there is an arc $\rho_t \in \widehat{R}(k)$, $t \in [-\epsilon, \epsilon]$, through ρ_0 such that* $\operatorname{tr}\rho_{-\epsilon}(m) < 2\cos\alpha$ *and* $\operatorname{tr}\rho_\epsilon(m) > 2\cos\alpha$.

Proof. The proof of the corollary can be derived from the proof of proposition 3.8 (see [Heu97, Theorem 5.10] for details). □

Corollary 1.3 is an immediate consequence of corollary 3.9.

4. The computation of $h^\alpha(k)$

The aim of this section is to compute the invariant $h^\alpha(k)$. We will only treat the case $\alpha \neq \pi/2$ but it is easy to see that the "exception" $\alpha = \pi/2$ is included as a limit case (see remark 4.9).

The reference for this section is Lin's paper (see [Lin92]). We restrict ourself to the points which are different from the result proved by Lin. It should be remarked that the lemma 4.4 simplifies the proof. Lemma 4.4 applies also in the case $\alpha = \pi/2$ and simplifies also Lin's proof.

The diploma thesis of the second author contains the main part of this section together with a more detailed discussion and many explicit examples (see [Kro96]).

The following lemma shows that the trace-free case is somehow special.

Lemma 4.1. *Let $\alpha \in (0, \pi)$ be given. Then*

(i): The space $\widehat{H}_2^{\pi/2}$ is a 2-sphere with four cone points deleted.

(ii): For $\alpha \neq \pi/2$ the space $\widehat{H}_2^\alpha$ is a 2-sphere with three points deleted.

Proof. Let $(\alpha, Q_i) \in \operatorname{Sp}(1)$, $i = 1, \ldots, 3$ be given. There exists an element $(\alpha, Q_4) \in \operatorname{Sp}(1)$ such that $(\alpha, Q_1)(\alpha, Q_2) = (\alpha, Q_3)(\alpha, Q_4)$ if and only if

$$\Re((\alpha, Q_3)^{-1}(\alpha, Q_1)(\alpha Q_2)) = \cos\alpha.$$

An easy calculation shows that this equation is equivalent to

$$\gamma(1+\langle Q_1,Q_2\rangle) = \langle Q_1\times Q_2+\gamma(Q_1+Q_2),Q_3\rangle \tag{2}$$

where $\gamma=\cot\alpha$. After conjugating we can assume that $Q_2=\mathbf{i}$ and $Q_1=\cos\theta_1\,\mathbf{i}+\sin\theta_1\mathbf{j}$ where $0\le\theta_1\le\pi$.

If $\alpha=\pi/2$ the equation (2) reduces to $\langle\sin\theta_1\,\mathbf{k},Q_3\rangle=0$. If $\sin\theta_1\ne0$ this implies $Q_3=\cos\theta_2\,\mathbf{i}+\sin\theta_2\mathbf{j}$ where $0\le\theta_2\le2\pi$. If $\sin\theta_1=0$ we obtain by conjugation $Q_3=\cos\theta_2\,\mathbf{i}+\sin\theta_2\mathbf{j}$ where $0\le\theta_2\le\pi$. So, parameterized by θ_1 and θ_2, $\widehat{H}_2^{\pi/2}$ is a pillowcase. The diagonal $\widehat{\Lambda}_2^{\alpha}$ is parameterized by (θ,θ), $0<\theta<\pi$.

The conjugacy classes of abelian representations $S_2^{\pi/2}$ are represented by the four points $A:=(\mathbf{i},\mathbf{i},\mathbf{i},\mathbf{i})$, $A':=(\mathbf{i},\mathbf{i},-\mathbf{i},-\mathbf{i})$, $B:=(\mathbf{i},-\mathbf{i},-\mathbf{i},\mathbf{i})$ and $B':=(\mathbf{i},-\mathbf{i},\mathbf{i},-\mathbf{i})$. Therefore, the four deleted points are given by the parameters $A=(0,0)$, $A'=(0,\pi)$, $B=(\pi,0)$ and $B'=(\pi,\pi)$.

If $\alpha\ne\pi/2$ we set

$$Q' := Q_1\times Q_2+\gamma(Q_1+Q_2),$$

$Q(\alpha,\theta_1):=Q'/\|Q'\|\in S^2$ and $c(\alpha,\theta_1):=\frac{\gamma}{\|Q'\|}(1+\langle Q_1,Q_2\rangle)$. More explicit we have

$$Q(\alpha,\theta_1) = -\frac{\sin(\frac{\theta_1}{2})\mathbf{k}}{(\sin^2(\frac{\theta_1}{2})+\gamma^2)^{1/2}}+\frac{\gamma(\cos(\frac{\theta_1}{2})\mathbf{i}+\sin(\frac{\theta_1}{2})\mathbf{j})}{(\sin^2(\frac{\theta_1}{2})+\gamma^2)^{1/2}}$$

and

$$c(\alpha,\theta_1) = \frac{\gamma\cos(\frac{\theta_1}{2})}{(\sin^2(\frac{\theta_1}{2})+\gamma^2)^{1/2}}.$$

Note that $Q(\alpha,\theta_1)$ lies on the right bisector of the geodesic segment between Q_1 and Q_2. Moreover, $|c(\alpha,\theta_1)|\le1$ and

$$|c(\alpha,\theta_1)| = \begin{cases} 1 & \Leftrightarrow \theta_1=0\\ 0 & \Leftrightarrow \theta_1=\pi.\end{cases} \tag{3}$$

It is obvious that the set

$$E:=E(\alpha,\theta_1):=\{X\in\mathbb{E}^3 \mid \langle Q(\alpha,\theta_1),X\rangle = c(\alpha,\theta_1)\}$$

is a plane orthogonal to $Q(\alpha,\theta_1)$ and that $c(\alpha,\theta_1)$ is the oriented distance from the origin. We see that the points Q_3 which satisfies equation (2) are exactly those in the intersection $S^2\cap E$. The set $S^2\cap E$ is in general a small circle on the sphere S^2. This small circle degenerates to a point if and only if $\theta_1=0$ and to a great circle if and only if $\theta_1=\pi$ (see equation (3)).

We set $\epsilon(\alpha) := \operatorname{sig} c(\alpha, \theta_1) = \operatorname{sig} \gamma \in \{\pm 1\}$. For a given $Q_3 \in S^2 \cap E$ we can write

$$Q_3 = c(\alpha, \theta_1) \cdot Q(\alpha, \theta_1) + \cos\theta_2 V_1 + \sin\theta_2 V_2$$

where $V_1 := V_1(\alpha, \theta_1) := Q_1 - c(\alpha, \theta_1) \cdot Q(\alpha, \theta_1)$ and $V_2 =: V_2(\alpha, \theta_1) := \epsilon(\alpha) \cdot Q \times V_1$ (note that $Q_1, Q_2 \in S^2 \cap E$). So, θ_2 is the oriented angle between Q_1 and Q_3 on $S^2 \cap E$.

The parameterization degenerates when $Q_1 = Q_2$ which is equivalent to $\theta_1 = 0$. Furthermore, the representations corresponding to the parameters (π, θ_2) and $(\pi, 2\pi - \theta_2)$ are equivalent (see figure 1).

The conjugacy classes of abelian representations $S_2^\alpha \cap H_2^\alpha$ are represented by the three points $(\mathbf{e}^{\mathbf{i}\alpha}, \mathbf{e}^{\mathbf{i}\alpha}, \mathbf{e}^{\mathbf{i}\alpha}, \mathbf{e}^{\mathbf{i}\alpha})$, $(\mathbf{e}^{\mathbf{i}\alpha}, \mathbf{e}^{-\mathbf{i}\alpha}, \mathbf{e}^{-\mathbf{i}\alpha}, \mathbf{e}^{\mathbf{i}\alpha})$ and $(\mathbf{e}^{\mathbf{i}\alpha}, \mathbf{e}^{-\mathbf{i}\alpha}, \mathbf{e}^{\mathbf{i}\alpha}, \mathbf{e}^{-\mathbf{i}\alpha})$. Therefore, the three deleted points are given by the parameters $A = (0, \theta_2)$, $B = (\pi, \pi)$ and $B' = (\pi, 0)$ where $0 \le \theta_2 \le 2\pi$. The diagonal $\widehat{\Lambda}_2^\alpha$ is parameterized by $(\theta, 0)$, $0 < \theta < \pi$, connecting the points A and B'. □

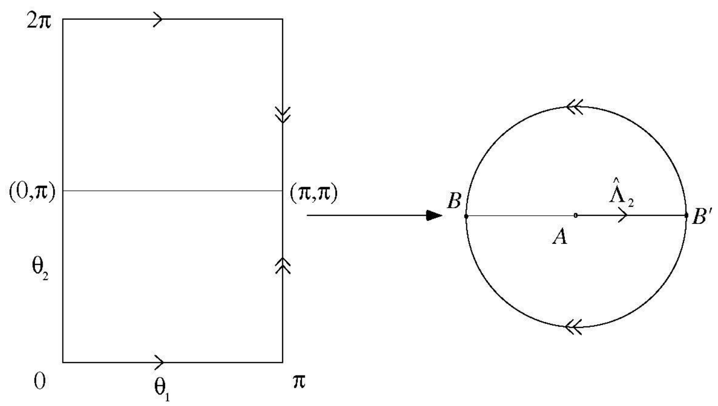

Figure 1.
Parameterization of $\widehat{H}_2^\alpha$

For the braid $\sigma_1 \in \mathfrak{B}_2$ we obtain $h^\alpha(\sigma_1) = 0$ because every point of $\Lambda_2^\alpha \cap \Gamma_{\sigma_1}^\alpha$ represents an abelian representation. Therefore the invariant vanishes for the trivial knot.

In order to determine the invariant for every knot we would like to study its behavior under crossing changes. Henceforth, let $\sigma \in \mathfrak{B}_n$ be a braid such that $k := \sigma^\wedge \subset S^3$ is a knot. Then $k' := (\sigma_1^2\sigma)^\wedge \subset S^3$ is also a knot and we now study the difference

$$\begin{aligned} h^\alpha(\sigma_1^2\sigma) - h^\alpha(\sigma) &= \langle \widehat{\Lambda}_n^\alpha, \widehat{\Gamma}_{\sigma_1^2\sigma}^\alpha \rangle - \langle \widehat{\Lambda}_n^\alpha, \widehat{\Gamma}_\sigma^\alpha \rangle \\ &= \langle \widehat{\Gamma}_{\sigma_1^{-2}}^\alpha, \widehat{\Gamma}_\sigma^\alpha \rangle - \langle \widehat{\Lambda}_n^\alpha, \widehat{\Gamma}_\sigma^\alpha \rangle \\ &= \langle \widehat{\Gamma}_{\sigma_1^{-2}}^\alpha - \widehat{\Lambda}_n^\alpha, \widehat{\Gamma}_\sigma^\alpha \rangle. \end{aligned}$$

Here we have chosen $\alpha \in (0,\pi)$, $\alpha \neq \pi/2$ such that $\Delta_k(e^{2\mathbf{i}\alpha}) \neq 0$ and $\Delta_{k'}(e^{2\mathbf{i}\alpha}) \neq 0$.

Observe that the "difference cycle" $(\Gamma_{\sigma_1^{-2}}^\alpha - \Lambda_n^\alpha)$ is carried by

$$V_n^\alpha := \{(A_1,\dots,A_n,B_1,\dots,B_n) \in H_n^\alpha \mid A_i = B_i,\, i=3,\dots,n\}.$$

Then $\widehat{V}_n^\alpha := (V_n^\alpha \setminus S_n^\alpha)/\mathrm{SO}(3)$ is a $(2n-2)$-dimensional submanifold of $\widehat{H}_n^\alpha$. Note that there is a natural projection $p\colon V_n^\alpha \to H_2^\alpha$ given by

$$p\colon (A_1,A_2,A_3,\dots,A_n,B_1,B_2,A_3,\dots,A_n) \mapsto (A_1,A_2,B_1,B_2).$$

To obtain a map between $\widehat{V}_n^\alpha$ and $\widehat{H}_2^\alpha$ we have to consider the set $W_n^\alpha := p^{-1}(S_2^\alpha \cap H_2^\alpha)$. It is easy to see that $W_n^\alpha \supset \Lambda_n^\alpha \cap \Gamma_{\sigma_1^{-2}}^\alpha$ is a $(2n-2)$-dimensional manifold. Therefore p induces a map $\widehat{p}\colon \widehat{V}_n^\alpha \setminus \widehat{W}_n^\alpha \to \widehat{H}_2^\alpha$ where $\dim \widehat{W}_n^\alpha = 2n-5$.

We apply the process used in [Lin92]: we perturb $\widehat{\Gamma}_\sigma^\alpha$ to $\widetilde{\Gamma}_\sigma^\alpha$ with compact support such that

$$(\widehat{\Gamma}_{\sigma_1^{-2}}^\alpha - \widehat{\Lambda}_n^\alpha) \pitchfork \widetilde{\Gamma}_\sigma^\alpha$$

which means precisely that $\widetilde{\Gamma}_\sigma^\alpha \pitchfork \widehat{\Gamma}_{\sigma_1^{-2}}^\alpha, \widehat{\Lambda}_n^\alpha$ and $\widetilde{\Gamma}_\sigma^\alpha \cap \widehat{W}_n^\alpha = \emptyset$. Moreover, we extend the isotopy so that $\widehat{V}_n^\alpha \pitchfork \widetilde{\Gamma}_\sigma^\alpha$. Thus $\widehat{V}_n^\alpha \cap \widetilde{\Gamma}_\sigma^\alpha$ is a 1-dimensional manifold.

We choose the orientations as in [Lin92]. It is sufficient to project the high dimensional manifolds via $\widehat{p}$ in order to study the difference $h^\alpha(\sigma_1^2\sigma) - h^\alpha(\sigma)$. More precisely, the following equation holds

$$h^\alpha(\sigma_1^2\sigma) - h^\alpha(\sigma) = \langle \widehat{\Gamma}_{\sigma_1^{-2}}^\alpha - \widehat{\Lambda}_2^\alpha, \widehat{p}\,(\widetilde{\Gamma}_\sigma^\alpha) \rangle_{\widehat{H}_2^\alpha}\,. \tag{4}$$

To study the intersection of $(\widehat{\Gamma}_{\sigma_1^{-2}}^\alpha - \widehat{\Lambda}_2^\alpha)$ with $\widehat{p}\,(\widetilde{\Gamma}_\sigma^\alpha)$ we have to understand the limiting behavior of the set $\widehat{p}\,(\widehat{\Gamma}_\sigma^\alpha)$ near the point A. In order to do this we consider the path $\gamma\colon \mathbb{R} \to H_2^\alpha$ given by

$$\gamma(t) = ((\alpha, Q_1(t)), (\alpha, Q_2), (\alpha, Q_3(t)), (\alpha, Q_4(t)))$$

where $Q_1(t) = \cos(\theta_1(t))\mathbf{i} + \sin(\theta_1(t))\mathbf{j}$, $Q_2 = \mathbf{i}$ and $Q_3(t) = c(t)Q(t) + \cos(\theta_2(t))$ $V_1(t) + \sin(\theta_2(t))V_2(t)$. Here $c(t) := c(\alpha,\theta_1(t))$, $Q(t) := Q(\alpha,\theta_1(t))$ and $V_i(t)$ are defined as in lemma 4.1. Note that Q_4 is determined by Q_1, Q_2 and Q_3. We choose $\theta_1(t)$ such that $\theta_1(0) = 0$ and $\theta_1^0 := (d\theta_1(t)/dt)_{t=0} \neq 0$. This implies $\gamma(0) = (\mathbf{e}^{\mathbf{i}\alpha}, \mathbf{e}^{\mathbf{i}\alpha}, \mathbf{e}^{\mathbf{i}\alpha}, \mathbf{e}^{\mathbf{i}\alpha})$ and γ gives a smooth path $\widehat{\gamma}\colon (-\epsilon, 0) \to \widehat{H}_2^\alpha$ for a small $\epsilon > 0$.

We call $\theta_1^0 := \theta_1^0(\widehat{\gamma}) := (d\theta_1(t)/dt)_{t=0}$ (resp. $\theta_2^0 := \theta_2^0(\widehat{\gamma}) := \theta_2(0)$) the *velocity* (resp. *angle*) of $\widehat{\gamma}$ in A.

The derivative $(dQ_3(t)/dt)_{t=0} \cdot (-\mathbf{j}) \in \mathbb{C}$ can be written in the form $\theta_1^0 \cdot s(\alpha, \theta_2^0)$ where

$$\begin{aligned} s(\alpha, \theta_2^0) &= z_\alpha + \cos(\theta_2^0)(1 - z_\alpha) + \sin(\theta_2^0)(1 - z_\alpha)\mathbf{i} \\ &= z_\alpha + \frac{1}{2\cos\alpha} e^{\mathbf{i}(\theta_2^0 + \alpha)} \\ &= \frac{\cos(\frac{2\alpha + \theta_2^0}{2})}{\cos\alpha} e^{\mathbf{i}\theta_2^0/2}. \end{aligned} \tag{5}$$

Here $z_\alpha := \frac{1}{2} - \frac{\mathbf{i}}{2}\tan\alpha \in \mathbb{C}$. By making use of the identification of $T_{\mathbf{e}^{\mathbf{i}\alpha}}(\Sigma_\alpha)$ with $\mathbb{C}$ we obtain

$$\gamma'(t) \cdot (-\mathbf{j}) = \theta_1^0 \sin\alpha\, (1, 0, s(\alpha, \theta_2^0), e^{-2\mathbf{i}\alpha}(1 - s(\alpha, \theta_2^0)) \in \mathbb{C}^4.$$

Note that $\theta_2^0(\widehat{\gamma})$ does not depend on the parameterization of $\widehat{\gamma}$.

Example 4.2. The angle of $\widehat{\Lambda}_2^\alpha$ in A is 0 and the angle of $\widehat{\Gamma}^\alpha_{\sigma_1^{-2}}$ in A is -4α. The latter can be seen as follows: by example 3.1 we have to look at

$$\begin{aligned} \frac{d}{dt}((\alpha, Q_2)^{-1} Q_1(t)(\alpha, Q_2))_{t=0} &= \frac{d}{dt}((\alpha, \mathbf{i})^{-1}(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})(\alpha, \mathbf{i}))_{t=0} \\ &= (\alpha, \mathbf{i})^{-1}(\mathbf{j})(\alpha, \mathbf{i}) \\ &= e^{-2\mathbf{i}\alpha}\mathbf{j}. \end{aligned}$$

This gives $\theta_2^0(\widehat{\Gamma}^\alpha_{\sigma_1^{-2}}) = -4\alpha$ (see figure 3). Therefore, $\widehat{\Gamma}^\alpha_{\sigma_1^{-2}}$ is an arc in $\widehat{H}_2^\alpha$ connecting A and B'. It is easy to see that $\widehat{\Gamma}^\alpha_{\sigma_1^{-2}} \cap \widehat{\Lambda}_2^\alpha = \emptyset$ and since $\widehat{\Gamma}^\alpha_{\sigma_1^{-2}}$ is a graph, it intersects each circle $\theta_2 \equiv$ constant in $\widehat{H}_2^\alpha$ in exactly one point (see figure 2). Moreover, $\widehat{\Gamma}^\alpha_{\sigma_1^{-2}}$ runs into B' as in the left hand side of figure 2 if $\pi/2 < \alpha < \pi$. The right hand side of figure 2 reflects the situation $0 < \alpha < \pi/2$. More precisely, let $\gamma\colon [0, \pi] \to \widehat{\Gamma}^\alpha_{\sigma_1^{-2}}$ be the parameterization of $\widehat{\Gamma}^\alpha_{\sigma_1^{-2}}$, $\gamma(t) := te^{\mathbf{i}\theta_2(t)}$, where $\theta_2(0) = -4\alpha$ and $\theta_2(\pi) = 2\pi$. A calculation as in equation (5) gives $(d\theta_2(t)/dt)_{t=\pi} = 2\epsilon(\alpha)\sin\alpha$ where $\epsilon(\alpha)$ is defined as in lemma 4.1.

Let K_α be the circle in $\mathbb{C}$ with center z_α and radius $1/(2\cos\alpha)$. Then $s(\alpha, \theta_2^0) \in K_\alpha$ in particular we have $0, 1, e^{-2\mathbf{i}\alpha} \in K_\alpha$.

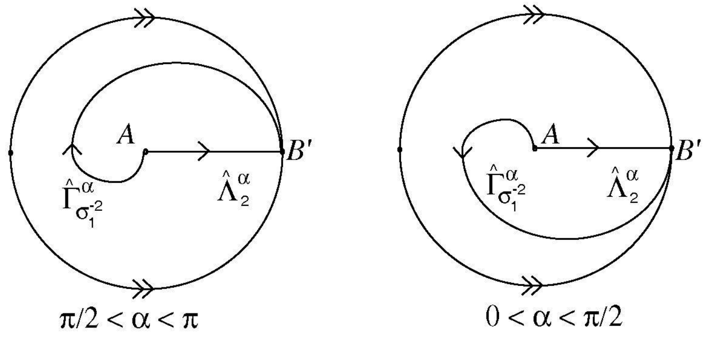

Figure 2.
The difference cycle $(\widehat{\Gamma}^{\alpha}_{\sigma_1^{-2}} - \widehat{\Lambda}^{\alpha}_2)$.

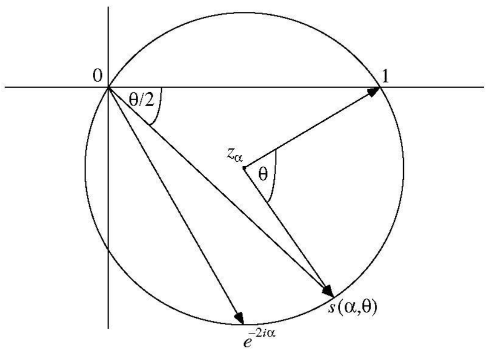

Figure 3.
The circle $K_\alpha \subset \mathbb{C}$.

Equation (5) tells us that for a given $s \in K_\alpha$ the θ_2-parameter of s is determined by the argument $\arg s$ where $0 \leq \arg s < 2\pi$. More precisely we have $\theta_2 = 2 \arg s$ (see figure 3). Note that $\theta_2(0) \equiv \pi - 2\alpha \bmod 2\pi$ is well defined.

Later we will make use of the following observation:

Lemma 4.3. *Let $s \in K_\alpha$ be given. Then $f_\alpha(s) := \frac{e^{2\,\alpha} s - 1}{s-1}$ is a real number.*

More precisely, $f_\alpha^{-1}(\mathbb{R}_{<0}) \subset K_\alpha$ is the open arc between 1 and $e^{-2i\alpha}$ which does not contain the 0.

Proof. The Moebius transformation

$$f_\alpha\colon z \mapsto \frac{e^{2\mathbf{i}\alpha}z-1}{z-1}$$

maps K_α into the real line. The conclusion of the lemma follows because $f_\alpha(0)=1$, $f_\alpha(e^{-2\mathbf{i}\alpha})=0$ and $f_\alpha(1)=\infty$ (see figure 3). □

Let $\mathbf{a} := (\mathbf{e}^{2\mathbf{i}\alpha},\dots,\mathbf{e}^{2\mathbf{i}\alpha}) \in S_n^\alpha$ be given. The point $\mathbf{a}$ is fixed by $\sigma\colon R_n^\alpha \to R_n^\alpha$. The derivative of σ at $\mathbf{a}$ is given by the Burau-matrix. More precisely we have the following commutative diagram (see [Lon89] and [Lin92])

$$\begin{array}{ccc} T_{\mathbf{a}}(R_n^\alpha) & \xrightarrow{\cdot(-\mathbf{j},\dots,-\mathbf{j})} & \mathbb{C}^n \\ {\scriptstyle D\ (\sigma)}\downarrow & & \downarrow{\scriptstyle \Phi^\sigma(e^{2\ \alpha})} \\ T_{\mathbf{a}}(R_n^\alpha) & \xrightarrow{\cdot(-\mathbf{j},\dots,-\mathbf{j})} & \mathbb{C}^n. \end{array}$$

We write $\Phi^\sigma(e^{2\mathbf{i}\alpha})$ in the form

$$\Phi^\sigma(e^{2\mathbf{i}\alpha}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$$

where $\mathbf{D}(e^{2\mathbf{i}\alpha})$ is a $(n-2)\times(n-2)$ matrix.

Lemma 4.4. *Let $\sigma \in \mathfrak{B}_n$ be given such that $k=\sigma^\wedge$ is a knot. Moreover, assume that $\alpha \in (0,\pi)$ is given such that $e^{2\mathbf{i}n\alpha} \neq 1$.*

If $\Delta_k(e^{2\mathbf{i}\alpha}) \neq 0$ then $\det(\mathbf{D}(e^{2\mathbf{i}\alpha})-\mathbf{E}) \neq 0$.

Proof. The normalized Alexander polynomial of $k := \sigma^\wedge$ is given by

$$\Delta_k(t) = (-\frac{1}{\sqrt{t}})^{e-n+1}\frac{1-t}{1-t^n}\det(\mathbf{E}-\phi^\sigma(t)) \tag{6}$$

where e is the exponent sum of σ (see [Jon87]). Note that $\Delta_k(t)=\Delta_k(t^{-1})$ and $\Delta_k(1)=1$. From equation (6) and lemma 2.2 the following correspondence can be derived

$$\Delta_k(t) = (-1)^e(\frac{1}{\sqrt{t}})^{e-n+1}c_{1,1}^\sigma(t). \tag{7}$$

Assume $\det(\mathbf{D}(e^{2\mathbf{i}\alpha})-\mathbf{E})=0$ then by lemma 2.2 we have $c_{1,1}^{\sigma_1^2\sigma}(e^{2\mathbf{i}\alpha}) = e^{4\mathbf{i}\alpha}c_{1,1}^\sigma(e^{2\mathbf{i}\alpha})$.

From this we deduce the formula

$$\Delta_{k'}(e^{2\mathbf{i}\alpha}) = e^{2\mathbf{i}\alpha}\cdot\Delta_k(e^{2\mathbf{i}\alpha}).$$

This gives a contradiction if $e^{2\mathbf{i}\alpha} \neq -1$ because $\Delta_k(e^{2\mathbf{i}\alpha})$, $\Delta_{k'}(e^{2\mathbf{i}\alpha}) \in \mathbb{R}$ and $e^{2\mathbf{i}\alpha} \notin \mathbb{R}$.

If $e^{2\mathbf{i}\alpha} = -1$ then we have $\Delta_k(-1) = -\Delta_{k'}(-1)$. Now k and k' differ only by a crossing change and therefore

$$\Delta_k(-1) - \Delta_{k'}(-1) = 2\mathbf{i}\Delta_{k''}(-1) \Longrightarrow \Delta_k(-1) = \mathbf{i}\Delta_{k''}(-1) \tag{8}$$

where $k'' := (\sigma_1\sigma)^\wedge$ is a 2-component link. Let $\widetilde{\Delta}_{k''}(t)$ be the Hosokawa polynomial of k''. The connection with the Alexander polynomial is given by

$$\Delta_{k''}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})\widetilde{\Delta}_{k''}(t) \tag{9}$$

(see [Mur96]). Since the Hosokawa polynomial is a symmetric integer polynomial we obtain from (8) and (9):

$$\Delta_k(-1) = \mathbf{i}\Delta_{k''}(-1) = -2\widetilde{\Delta}_{k''}(-1).$$

Therefore we have $\Delta_k(-1) \equiv 0 \bmod 2$ which is impossible. □

As in the trace-free case we investigate the projection $\widehat{p}(\widehat{\Gamma}^\alpha_\sigma)$ in a neighborhood of A (see [Lin92]).

Lemma 4.5. *Choose $n \in \mathbb{N}$ such that $e^{2\mathbf{i}n\alpha} \neq 1$. Then in a neighborhood of A on $\widehat{H}^\alpha_2$, $\widehat{p}(\widehat{\Gamma}^\alpha_\sigma \cap \widehat{V}^\alpha_n)$ is a curve approaching A.*

Moreover, the angle θ^0_2 of $\widehat{p}(\widehat{\Gamma}^\alpha_\sigma \cap \widehat{V}^\alpha_n)$ at A is not equal to 0 or -4α.

Proof. The main steps of the proof are analogous to Lin's proof (see [Lin92, Lemma 2.4]). But by lemma 4.4 we have only to consider **Case 1** which simplifies the trails of the proof.

Let θ^0_2 be the angle of $\widehat{p}(\widehat{\Gamma}^\alpha_\sigma \cap \widehat{V}^\alpha_n)$ at A. We have $\theta^0_2 = 2\arg s_\sigma$ where the parameter s_σ is given by

$$\Phi^\sigma(e^{2\mathbf{i}\alpha})\begin{pmatrix}1\\0\\\mathbf{v}\end{pmatrix} = \begin{pmatrix}s_\sigma\\e^{-2\mathbf{i}\alpha}(1-s_\sigma)\\\mathbf{v}\end{pmatrix}.$$

Here

$$\mathbf{v} = (\mathbf{E}-\mathbf{D})^{-1}\mathbf{C}\begin{pmatrix}1\\0\end{pmatrix}.$$

Since we have chosen $e^{2\mathbf{i}\alpha}$ such that $\Delta_k(e^{2\mathbf{i}\alpha}) \neq 0 \neq \Delta_{k'}(e^{2\mathbf{i}\alpha})$ the second statement of the lemma 4.5 follows from the same conclusions as Lin's proof. □

Lemma 4.6. *$h^\alpha(\sigma_1^2\sigma) - h^\alpha(\sigma) = \epsilon$, where*

$$\epsilon = \begin{cases} 0 & \text{if } s_\sigma \in f_\alpha^{-1}(\mathbb{R}_{>0}) \\ 1 & \text{if } s_\sigma \in f_\alpha^{-1}(\mathbb{R}_{<0}). \end{cases}$$

Proof. We start by a perturbation $\widehat{\Gamma}_\sigma^\alpha \rightsquigarrow \breve{\Gamma}_\sigma^\alpha$ with compact support such that $\breve{\Gamma}_\sigma^\alpha \pitchfork (\widehat{\Gamma}_{\sigma_1^{-2}}^\alpha - \widehat{\Lambda}_n^\alpha)$. We claim that there is a neighborhood of $B' = (\pi, 0)$ on $\widehat{H}_2^\alpha$ such that $\widehat{p}(\breve{\Gamma}_\sigma^\alpha \cap \widehat{V}_n^\alpha)$ is disjoint with that neighborhood. Suppose this is not true. Then we will get a point in $\breve{\Gamma}_\sigma^\alpha \cap \widehat{V}_n^\alpha$ which is represented by $(\mathbf{A}, \mathbf{A})$ where $\mathbf{A} := (\mathbf{e}^{2\mathbf{i}\alpha}, \mathbf{e}^{-2\mathbf{i}\alpha}, A_3, \dots, A_n)$. But $(\mathbf{A}, \mathbf{A})$ represents also a point of $\widehat{W}_n^\alpha$. This gives a contradiction if $(\mathbf{A}, \mathbf{A})$ represents an irreducible point (remember: $\widehat{W}_n^\alpha \cap \breve{\Gamma}_\sigma^\alpha = \emptyset$). If $(\mathbf{A}, \mathbf{A})$ represents an reducible point then this point is already in $\Gamma_\sigma^\alpha \cap V_n^\alpha$ since the isotopy above has compact support. Therefore, $\mathbf{A}$ would be a fixed point of $\sigma\,|_{R_n^\alpha}$. But this is impossible since the permutation induced by σ is a n-cycle.

We continue by a small perturbation relative to a neighborhood of the reducible point $(\mathbf{e}^{\mathbf{i}\alpha}, \dots, \mathbf{e}^{\mathbf{i}\alpha})$ changing $\breve{\Gamma}_\sigma^\alpha$ to $\widetilde{\Gamma}_\sigma^\alpha$ such that $\widetilde{\Gamma}_\sigma^\alpha \pitchfork \widehat{V}_n^\alpha$. Then $\widehat{p}(\widetilde{\Gamma}_\sigma^\alpha \cap \widehat{V}_n^\alpha)$ is a 1-dimensional submanifold of $\widehat{H}_2^\alpha$. In a neighborhood of A, it is a curve approaching A with angle $\theta_2^0 = 2\arg s_\sigma$. The other end of the curve must approach B. Since

$$h^\alpha(\sigma_1^2\sigma) - h^\alpha(\sigma) = \langle \widehat{\Gamma}_{\sigma_1^{-2}}^\alpha - \widehat{\Lambda}_2, \widehat{p}(\widetilde{\Gamma}_\sigma^\alpha)\rangle_{\widehat{H}_2^\alpha}$$

and the angle of $\widehat{\Lambda}_\sigma^\alpha$ (resp. $\widehat{\Gamma}_{\sigma_1^{-2}}^\alpha$) at A is 0 (resp. -4α) the conclusion of the lemma follows (see figure 2). □

Lemma 4.7. *Under the assumptions of lemma 4.5 we have*

$$\frac{\Delta_{k'}(e^{2\mathbf{i}\alpha})}{\Delta_k(e^{2\mathbf{i}\alpha})} = \frac{e^{2\mathbf{i}\alpha}s_\sigma - 1}{s_\sigma - 1} = f_\alpha(s_\sigma).$$

Proof. The proof is completely analogous to Lin's proof and again the fact $\det(\mathbf{D} - \mathbf{E}) \neq 0$ simplifies the argument. □

We are now able to state the main result of this section

Proposition 4.8. *Let $k_- = (\sigma_1^2\sigma)^\wedge$ and $k_+ = (\sigma)^\wedge$ be knots. Moreover, let $\alpha \in (0, \pi)$ be given such that $\Delta_{k_+}(e^{2\mathbf{i}\alpha}) \neq 0 \neq \Delta_{k_-}(e^{2\mathbf{i}\alpha})$. Then*

$$h^\alpha(k_-) - h^\alpha(k_+) = \begin{cases} 0 & \text{iff } \Delta_{k_+}(e^{2\mathbf{i}\alpha}) \cdot \Delta_{k_-}(e^{2\mathbf{i}\alpha}) > 0 \\ 1 & \text{iff } \Delta_{k_+}(e^{2\mathbf{i}\alpha}) \cdot \Delta_{k_-}(e^{2\mathbf{i}\alpha}) < 0. \end{cases}$$

Proof. Assume again that $e^{2\mathbf{i}n\alpha} \neq 1$.

Lemma 4.7 gives us

$$f_\alpha(s_\sigma) > 0 \text{ iff } \Delta_{k_-}(e^{2\mathbf{i}\alpha}) \cdot \Delta_{k_+}(e^{2\mathbf{i}\alpha}) > 0$$

and

$$f_\alpha(s_\sigma) < 0 \text{ iff } \Delta_{k_-}(e^{2\mathbf{i}\alpha}) \cdot \Delta_{k_+}(e^{2\mathbf{i}\alpha}) < 0$$

where f_α is the Moebius transformation defined in lemma 4.3. The result follows now from lemma 4.6. □

There is also a recursive procedure for calculating the value of the signature function. Let $\omega \in S^1$ such that $\Delta_k(\omega) \neq 0$. The signature $\sigma_k(\omega)$ is always an even integer. We have

$$\sigma_k(\omega) \equiv 0 \bmod 4 \text{ iff } \Delta_k(\omega) > 0 \quad \text{and} \quad \sigma_k(\omega) \equiv 2 \bmod 4 \text{ iff } \Delta_k(\omega) < 0. \tag{10}$$

Moreover, for knots k_+ and k_- we obtain from simple consideration of the Seifert matrices of k_+ and k_-

$$0 \leq \sigma_{k_-}(\omega) - \sigma_{k_+}(\omega) \leq 2 \tag{11}$$

(see [Lip90] and [Gil82]).

We are ready to proof the main result of this section:

Proof of theorem 1.2. The invariants $h^\alpha(k)$ and $\sigma_k(e^{2\mathbf{i}\alpha})$ are defined and they are both locally constant. Choose an irrational angle $\beta \in (0,\pi)$, $\beta < \alpha$ such that $\Delta_k(e^{2\mathbf{i}t}) \neq 0$ for all $t \in [\beta,\alpha]$.

It follows that $h^\alpha(k) = h^\beta(k)$ and $\sigma_k(e^{2\mathbf{i}\alpha}) = \sigma_k(e^{2\mathbf{i}\beta})$. Since β is irrational we have $\Delta_{k'}(e^{2\mathbf{i}\beta}) \neq 0$ for every knot $k' \subset S^3$.

The inductive procedure given by proposition 4.8 and by equations (10) and (11) makes it possible to prove that

$$h^\beta(k) = \frac{1}{2}\sigma_k(e^{2\mathbf{i}\beta}).$$ □

Remark 4.9.

1. Note that theorem 1.2 includes the case $\alpha = \pi/2$ because $\Delta_k(-1) \neq 0$.
2. We are not able to calculate the value of $h^\alpha(k)$ if $\widehat{\Lambda}_n^\alpha \cap \widehat{\Gamma}_\sigma^\alpha$ is compact but $\Delta_k(e^{2\mathbf{i}\alpha}) = 0$.

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