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A character formula for a family of simple modular representations of GL_n

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Abstract. Let K be an algebraically closed field of finite characteristic p, and let $n \geq 1$ be an integer. In the paper, we give a character formula for all simple rational representations of $GL_n(K)$ with highest weight any multiple of any fundamental weight. Our formula is slightly more general: say that a dominant weight λ is *special* if there are integers $i \leq j$ such that $\lambda = \sum_{i \leq k \leq j} a_k \, \omega_k$ and $\sum_{i \leq k \leq j} a_k \leq \inf(p-(j-i), p-1)$. Indeed, we compute the character of any simple module whose highest weight λ can be written as $\lambda = \lambda_0 + p\lambda_1 + ... + p^r \lambda_r$ with all λ_i are special. By stabilization, we get a character formula for a family of irreducible rational $GL_\infty(K)$-modules.

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Introduction

In the paper, we will prove a character formula for a stable family of simple polynomial representations of $GL_n(K)$. Unfortunately, the main result of the paper requires some preparatory explanations. Therefore, the introduction is organized as follows. We first define the basic notions about polynomial weights and we describe some combinatorics involving Young diagrams. Next, we recall the usual correspondence between dominant polynomial weights and Young diagrams and we compare the corresponding definitions. After the statement of the main result, we explain the meaning of a stable family in terms of polynomial functors. Then, we briefly compare our result with the main result of [AJS] about Lusztig's Conjecture. At the end of the introduction, we describe the main ingredients of the proof which uses tilting modules [D2][R] and the modular Verlinde formula [GM1][GM2].

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Let us start with definitions involving weights. From now on, fix a positive integer n and an algebraically closed field K of characteristic $p > 0$. Let H be the Cartan subgroup of $GL_n(K)$ consisting of diagonal matrices and let P be the group of characters of H. An element of P will be called a *weight* and the group structure of P will be denoted additively. Denote by $\epsilon_1, \epsilon_2, \ldots$ the natural basis P, i.e. $\epsilon_i(h)$ is the i^{th} diagonal entry of the matrix $h \in H$. Therefore any weight μ can be written as $\mu = \sum_{1 \leq i \leq n} r_i\, \epsilon_i$, where $r_i \in \mathbb{Z}$. Its *degree* is $|\mu| = \sum_{1 \leq i \leq n} r_i$. The weight μ is called *polynomial* if $r_i \geq 0$ for all i. It is called *dominant* if $r_1 \geq r_2 \geq \ldots$. By definition, the k^{th}*-fundamental weight* is $\omega_k = \sum_{1 \leq i \leq k} \epsilon_k$, for any k with $1 \leq k \leq n$. Therefore a weight λ is polynomial and dominant if and only if $\lambda = \sum_{1 \leq k \leq n} a_k\, \omega_k$ where $a_k \geq 0$ for any k. The main definition of the paper is the definition of special weights. A dominant polynomial weight λ is *special* if and only if there exist integers $i \leq j$ such that:

(i) $\lambda = \sum_{i \leq k \leq j} a_k\, \omega_k$,

(ii) $m(\lambda) \leq p - (j - i)$ and $m(\lambda) < p$, where $m(\lambda) = \sum_{i \leq k \leq j} a_k$.

Note that the last inequality $m(\lambda) < p$ is automatically satisfied whenever $i \neq j$. We will also use the notion of the p-adic expansion of a polynomial weight. Recall that any integer $l \geq 0$ admits a unique p-adic expansion $l = \sum_{j \geq 0} l(j) p^j$, where $0 \leq l(j) < p$ for all $j \geq 0$ (this expansion is finite since $l(j) = 0$ for $j >> 0$). Similarly, any polynomial weight μ admits a unique finite *p-adic expansion* $\mu = \sum_{j \geq 0} p^j \mu(j)$, which is defined by $\mu(j) = \sum_{1 \leq i \leq n} r_i(j)\, \epsilon_i$. Also set $\mathcal{C}_n$ the set of all dominant polynomial weights λ of the form $\lambda = \sum_{k \geq 0} p^k \lambda_k$, where all weights λ_k are special and $\lambda_k = 0$ for $k >> 0$. Indeed, it is easy to see that $\sum_{k \geq 0} p^k \lambda_k$ is the p-adic expansion of λ, i.e. we have $\lambda_k = \lambda(k)$ for all $k \geq 0$ (see Lemma 5.1 (i)).

Now, we will define a few notions involving Young diagrams. The *degree* of a Young diagram Y, denoted by $|Y|$, is the total number of boxes and its *height* is the number of rows. A tableau of shape Y is a labeling of the boxes of Y by the integers $1, 2, \ldots, n$. It is convenient to draw Young diagrams and tableaux and the convention used in the paper is better explained by giving one example of a Young diagram Y of degree 8 and height 3 and one example of a tableau T of shape Y:

$$Y : \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & \\ \cline{1-3} \\ \cline{1-1} \end{array} \qquad\qquad T : \begin{array}{|c|c|c|c|} \hline 1 & 3 & 3 & 3 \\ \hline 2 & 4 & 5 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

As usual, a tableau is called *semi-standard* if the filling is non decreasing from left to right and increasing from top to bottom, e.g. the tableau in the previous example is semi-standard. For a tableau T, denote by $T[i]$ the subset of boxes with labels $\leq i$. Therefore, when T is semi-standard, $T[i]$ is again a Young diagram. The *weight* of T is $w(T) = \sum_{1 \leq i \leq n} \eta_i(T) \epsilon_i$, where $\eta_i(T)$ equals the number of times the label i occurs in T (i.e. $\eta_i(T)$ is the cardinality of $T[i] \setminus T[i-1]$). For a Young diagram Y, we denote by $c_i(Y)$ the number of boxes on the i^{th} column and by $r_j(Y)$ the number of boxes on the j^{th} row. In the previous example, $c_1(Y) = 3$, $r_1(Y) = 4$, $c_2(Y) = 2$, $r_2(Y) = 3$ and so on $\ldots$. Let $m < p$ be a positive integer. We say that Y is m-special if the number of columns is $\leq m$ and if $c_1(Y) - c_m(Y) \leq p - m$. By definition, a semi-standard tableau T is *m-semi-standard* if all Young diagrams $T[i]$ are m-special.

There is a one-to-one correspondence $\lambda \mapsto Y(\lambda)$ between dominant polynomial weights λ and Young diagrams of height $\leq n$. Indeed, $Y(\lambda)$ is defined by the requirement: $\lambda = \sum_{1 \leq i \leq n} r_i(Y)\epsilon_i$. This correspondence preserves the degree. Moreover a polynomial dominant weight λ is special if and only if $Y(\lambda)$ is $m(\lambda)$-special, see Lemma 4.1 (i). Let λ be a special weight, let μ be a polynomial weight. Set $N(\lambda, \mu)$ the number of $m(\lambda)$-semi-standard tableaux of shape $Y(\lambda)$ and weight μ. By definition, we have $N(0,0) = 1$ and $N(\lambda, \mu) = 0$ if the degrees of λ and μ are distinct.

For any dominant weight λ, set $V = K^n$ and let $L_V(\lambda)$ be the simple $GL(V)$-module with highest weight λ (this simple $GL_n(K)$-module is usually denoted by $L(\lambda)$). For $\mu \in P$, its weight space corresponding to the weight μ is denoted by $L_V(\lambda)_\mu$. The main result of the paper is the following:

Theorem 5.3. *Let $\lambda \in \mathcal{C}_n$. Any weight of $L_V(\lambda)$ is polynomial, and for any polynomial weight μ, we have:*

$$\dim L_V(\lambda)_\mu = \prod_{k \geq 0} N(\lambda(k), \mu(k)).$$

In the theorem, we stated the obvious fact that any weight μ of $L_V(\lambda)$ is polynomial because this property is necessary to define its p-adic expansion. Also, the infinite product is well defined because $N(\lambda(k), \mu(k)) = N(0,0) = 1$ for $k >> 0$.

It remains to explain what means a *stable* family of simple modules. The definition of special weights is indeed independent of n, i.e. if λ is a special weight for $GL_n(K)$ its natural extension to $GL_N(K)$ is again special for any $N \geq n$. Otherwise stated, the family $(\mathcal{C}_n)_{n \geq 1}$ is *stable*, i.e. $\mathcal{C}_n \subset \mathcal{C}_{n+1}$. Thus the previous theorem gives rise to a character formula for any simple $GL_\infty(K)$-module with highest weight $\lambda \in \mathcal{C}_\infty$, where $GL_\infty(K) = \cup_{n \geq 1} GL_n(K)$, $\mathcal{C}_\infty = \cup_{n \geq 1} \mathcal{C}_n$. The stability notion can be better explained in terms of polynomial functors. Let Y be a Young diagram and let λ be the corresponding polynomial dominant weight. It will be convenient to extend the notation $L_V(\lambda)$ by requiring $L_V(\lambda) = 0$ if λ is not a dominant weight for $GL_n(K)$, i.e. if the height of Y is $> n$. Then there is a polynomial functor S_Y such that $S_Y : V \mapsto L_V(\lambda)$ for all $n \geq 0$ (our definition of the functor S_Y is not complete, because we only describe the values of the functor on objects). Therefore, the previous theorem is indeed a character formula for any simple polynomial functor S_Y, where $Y = Y(\lambda)$ for some $\lambda \in \mathcal{C}_\infty$.

Example. For any s with $0 \leq s \leq p-1$, the weight $s\,\omega_i$ is special. Therefore, $N\,\omega_i \in \mathcal{C}_n$ for any $N \geq 0$, and the theorem gives a character formula for any simple module whose highest weight is a multiple of a fundamental weight.

There is a general conjecture, due to Lusztig [Lu1,Lu2], about the character of a simple rational $GL_n(K)$-module. The experts believe that this conjecture holds for $p \geq n$ (see e.g. the introduction of [So]) and it has been proved for $p >> n$ by Andersen, Jantzen and Soergel [AJS]. In contrast, our character formulas apply only to some peculiar highest weights, but they hold for any n and are therefore outside the validity domain of Lusztig's Conjecture. Indeed Lusztig's Conjecture does not seem adapted to the investigation of simple polynomial functors. Using Weyl's polarizations, the simple polynomial functor S_Y is entirely determined by the $GL_n(K)$-module $S_Y(K^n)$, where $n = |Y|$. Therefore, Lusztig's Conjecture only applies to polynomial functors of degree $\leq p$ and simple polynomial functors of degree $\leq p$ can be easily determined by elementary computations or by Theorem 5.3.

The proof is based on the following three ingredients:

(i) First, one uses Steinberg's tensor product formula [St] to reduce the statement to the case where λ is special. It turns out that Steinberg's formula is especially simple in our setting, because any weight of $L_V(\lambda)$ is a unique combination of weights of the modules $L_V(p^k\,\lambda(k))$ (Lemma 5.2).

(ii) We strongly use an idea of Donkin [D2]: Donkin proved that $\mathbb{M} = \bigwedge(V \otimes W)$ is a dual pair under $GL(V) \times GL(W)$ (here W is another vector space). This dual pair is called Howe's skew dual pair, because it has been found by Howe in the context of fields of characteristic zero [H]. Donkin showed that the character of all simple modules can be deduced from the character of all tilting modules, and conversely. However, we do not have such an information. This is why we need to modify a bit Donkin's approach. Using the same dual pair, we show that the character of simple $GL(V)$-modules can be also deduced from the tensor product mutiplicity of a given tilting $GL(W)$-module (Corollary 2.3) in some direct summands of the $GL(W)$-module $\mathbb{M}$.

(iii) Similarly, the general tensor product multiplicities of tilting modules are unknown. However the main result of [GM1,GM2] (Verlinde's formula for algebraic groups) describes some of them. More precisely, we use Verlinde's formula for $GL(W)$ with W of dimension $1, 2, \ldots, p-1$, and then the computable tilting multiplicities in $\mathbb{M}$ correspond exactly to the special weights (see e.g. Lemma 3.4).

Remark. It follows from the character formula that the restriction to $GL(n-1)$ of representations considered here are semi-simple (Theorem 6.2). This result has been obtained independently and simultaneously by J. Brundan, A. Kleshchev and I. Suprunenko [BKS] by very different methods. Indeed, the result of [BKS] is more precise, because it characterizes all simple representations of $GL(n)$ whose restrictions to $GL(n-1)$ are semi-simple. Later, these three authors have been able to recover the main result of our paper (Theorem 5.3) by using their semi-simplicity theorem (thus providing a very different proof).

1. General results about tilting modules

Let K be an algebraically closed field of characteristic p, let G be a reductive group over K, let B be a Borel subgroup, and let $H \subset B$ be a Cartan subgroup. We will set by U the unipotent radical of B and by U^- the unipotent radical of the opposed Borel subgroup. Denote by P^+ the set of dominant weights relative to B. For $\lambda \in P^+$, denote by $L(\lambda)$ (respectively $\Delta(\lambda)$, $\nabla(\lambda)$) the simple module (respectively the Weyl module, the dual of the Weyl module) with highest weight λ.

By G-module, we mean rational G-module of finite dimension. A good filtration of a G-module M is a filtration whose subquotients are dual of Weyl modules. A G-module M is tilting if M and M^* have a good filtration. Recall the following known result:

Theorem 1.1.
(i) For each $\lambda \in P^+$, there exists a unique indecomposable tilting module $T(\lambda)$ which admits λ as highest weight. Moreover, $\dim T(\lambda)_\lambda = 1$.
(ii) Any tilting module is the direct sum of indecomposable tilting modules of type $T(\lambda)$. The tilting modules $T(\lambda)$ and $T(\mu)$ are isomorphic if and only if $\lambda = \mu$.
(iii) The tensor product of two tilting modules is a tilting module.

References for the Theorem are as follows: the general notion of tilting modules for any quasi-hereditary algebra is due to Ringel [R]. In the context of algebraic groups, the assertions (i), (ii) are due to Donkin [D2] (Theorem 1.1). Assertion (iii) follows from the fact that the tensor product of two G-modules with a good filtration has a good filtration: for groups of type A (which are indeed the only groups used here), it has been established in [W], for the general case see [D1], [M1].

Let M be a G-module. Denote by $T^G(M)$ the image of the composite map $M^U \to M \to M_{U^-}$ where M^U is the space of U-invariants of M and $M_{U^-} = H_0(U^-, M)$ is the space of U^--coinvariants of M. Since $T^G(M)$ is an H-module, there is a weight decomposition $T^G(M) = \oplus_{\lambda \in P^+} T^G_\lambda(M)$.

Lemma 1.2. *Let M be an indecomposable tilting module.*
(i) $T^G(M)$ has dimension one.
(ii) Let λ be the unique weight of $T^G(M)$. Then we have $M \simeq T(\lambda)$.

Proof. It is clear that $T^G(N) \neq 0$ for any non-zero G-module N, because any maximal weight of N is a weight of $T^G(N)$. Let λ be any weight of $T^G(M)$ and choose $v \in M^U_\lambda$ such that its image in $T^G(M)$ is not zero. Denote by the same notation v_λ a highest weight vector in $\Delta(\lambda)$, in $\nabla(\lambda)$ and in $T(\lambda)$. By the universal property of Weyl modules, there is a map $\psi_1 : \Delta(\lambda) \to M$ sending v_λ to v. Similarly, there is a map $\psi_2 : M \to \nabla(\lambda)$ sending v to v_λ.

Now, there is a canonical injection $\Delta(\lambda) \hookrightarrow T(\lambda)$ (sending v_λ to v_λ) whose quotient has a filtration by Weyl modules. We have $Ext^1_G(\Delta(\mu), \nabla(\mu')) = 0$, for any $\mu, \mu' \in P^+$ ([CPSV], corollary 3.3). Since M has a good filtration, we have $Ext^1_G(T(\lambda)/\Delta(\lambda), M) = 0$. Thus, the map ψ_1 can be extended to a map $\phi_1 : T(\lambda) \to M$. In the same way, there is a canonical surjection $T(\lambda) \twoheadrightarrow \nabla(\lambda)$ (sending v_λ to v_λ), and the map ψ_2 can be lifted to a map $\phi_2 : M \to T(\lambda)$. So we get the following commutative diagram:

$$\begin{tikzcd} & & T(\lambda) \arrow[d] \\ \Delta(\lambda) \arrow[r, "\psi_1"] \arrow[d] & M \arrow[r, "\psi_2"] \arrow[ur, "\phi_2"] & \nabla(\lambda) \\ T(\lambda) \arrow[ur, "\phi_1"'] & & \end{tikzcd}$$

By definition, we have $\psi_2 \circ \psi_1(v_\lambda) = v_\lambda$. Therefore, $\phi_2 \circ \phi_1$ is a non nilpotent endomorphism of the indecomposable module M. By Fitting's Lemma, $\phi_2 \circ \phi_1$ is an invertible map. Thus, $T(\lambda)$ is a direct factor of M and so we have $M \simeq T(\lambda)$.

If ν is another weight of $T^G(M)$, we get $T(\nu) \simeq M \simeq T(\lambda)$. Therefore by Theorem 1.1 (ii), λ is the unique weight of $T^G(M)$. As $T(\lambda)_\lambda$ has dimension 1 (Theorem 1.1 (i)), it follows that $T^G(M)$ has dimension one. $\square$

Corollary 1.3. *Let M be a tilting G-module and let C be its commutant.*
(i) We have $M \simeq \oplus_{\lambda \in P^+} T^G_\lambda(M) \otimes T(\lambda)$ as a G-module.
(ii) For any $\lambda \in P^+$, the C-module $T^G_\lambda(M)$ is zero or simple.

Proof. By Theorem 1.1, there exists an isomorphism of G-modules $M \simeq \oplus_{\lambda \in P^+} T(\lambda)^{\otimes N_\lambda}$. By Lemma 1.2 we have $N_\lambda = dim T^G_\lambda(M)$ and Assertion (i) follows. For $N \geq 0$, denote by $Mat_N(K)$ be the K-algebra of $N \times N$ matrices. Clearly, C contains a subalgebra $C^0 \simeq \oplus_{\lambda \in P^+} Mat_{N_\lambda}(K)$ and we have $M \simeq \bigoplus T^G_\lambda(M) \otimes T(\lambda)$ as $C^0 \times G$-modules. Hence Assertion (ii) follows from the fact that for any λ, $T^G_\lambda(M)$ is zero or is a simple C^0-module. $\square$

Lemma 1.4. *Let M, N be two G-modules. If M is indecomposable of dimension divisible by p, then the dimension of any direct summand of $M \otimes N$ is divisible by p.*

Proof. This follows easily from Theorem 2.1 of [BC], see also [GM1] (Lemma 2.7.). $\square$

2. Howe's skew duality for the pair $(GL(V), GL(W))$

From now on, fix an integer $n \geq 1$ and set $V = K^n$. We need to modify some notations of the introduction. The Cartan subgroup of $GL(V)$ will be denoted by H_V (instead of H), the group of characters of H_V by P_V (instead of P), the basis elements of P_V by ϵ_i^V (instead of ϵ_i) and the fundamental weights by ω_k^V (instead of ω_k). We will also modify some notations of Section 1. The set of dominant weights will be denoted by P_V^+ and for a $\lambda \in P_V^+$, we will denote by $L_V(\lambda)$, $\nabla_V(\lambda)$ and $T_V(\lambda)$ the simple module, the dual of the Weyl module and the tilting module with highest weight λ. We will use the following additional notations. Let $(v_i)_{1\leq i\leq n}$ be the natural basis of $V = K^n$. Let U_V (respectively U_V^-) be the subgroup of unipotent upper diagonal (respectively lower diagonal) matrices. Indeed $P_V^+ = \oplus_{1\leq i<n}\mathbb{N}\,\omega_i^V \oplus \mathbb{Z}\omega_n^V$, and the dominant weights are relative to the Borel subgroup $H_V.U_V$.

In what follows, we will use another vector space W of dimension m, with basis $(w_i)_{1\leq i\leq m}$. Notations relative to $GL(W)$ will be similar to those for $GL(V)$.

For any Young diagram Y contained in the $n \times m$ rectangle (i.e. such that $c_1(Y) \leq n$ and $r_1(Y) \leq m$), we set $\lambda(Y) = \sum_{1\leq i\leq n} r_i(Y)\,\epsilon_i^V$ and $\lambda^T(Y) = \sum_{1\leq i\leq m} c_i(Y)\,\epsilon_i^W$. By definition, $\lambda(Y)$ belongs to P_V^+ and $\lambda^T(Y)$ belongs to P_W^+. The map $Y \mapsto \lambda(Y)$ is the inverse of the map $\lambda \mapsto Y(\lambda)$ defined in the introduction. Set $\mathbb{M} = \bigwedge(V \otimes W)$, let $K[GL(V)]$ be the group algebra of $GL(V)$ and let $\rho_V : K[GL(V)] \to End_K(\mathbb{M})$ the map induced by the action of $GL(V)$ on $\mathbb{M}$.

Theorem 2.1. (Donkin)

(i) We have $\rho_V(K[GL(V)]) = End_{GL(W)}(\mathbb{M})$.

(ii) As a $GL(W)$-module, $\mathbb{M}$ is tilting.

Proof. Theorem 2.1 (i) is proved in [D2], proposition (3.11) (see also [AR] for a generalization to other classical groups). As a $GL(W)$-modules, $\bigwedge W$ is titling (see [D2] or Lemma 3.2) and $\mathbb{M}$ is isomorphic to $(\bigwedge W)^{\otimes n}$. Therefore, by Theorem 1.1 (iii) the $GL(W)$-module $\mathbb{M}$ is tilting (see also [D2]). □

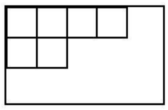
Indeed, we obtain dual statements by exchanging V and W. However, it should be noted that usually $\mathbb{M}$ is not tilting as a $GL(V) \times GL(W)$-module. In Howe's terminology, $(GL(V), GL(W))$ is a dual pair in $GL(\mathbb{M})$. Indeed, for fields of characteristic zero, this duality is due to Howe [H]. In this setting, Howe showed that the $GL(V) \times GL(W)$-module $\mathbb{M}$ is isomorphic to $\oplus_Y L_V(\lambda(Y)) \otimes L_W(\lambda^T(Y))$, where Y runs over all Young diagrams contained in the $n \times m$ rectangle. A certain generalization of this property in finite characteristics is stated in the next lemma:

Lemma 2.2. *Let Y be a Young diagram of degree d contained in the $n \times m$ rectangle.*

(i) As $GL(V)$-module, $T^{GL(W)}_{\lambda^T(Y)}(\mathbb{M})$ is isomorphic to $L_V(\lambda(Y))$.

(ii) Let $\mu = \sum_{1\leq k\leq n} k_i \epsilon_i^V$ *be a polynomial weight of degree d. We have* $L_V(\lambda(Y))_\mu \simeq T^{GL(W)}_{\lambda^T(Y)}(\bigwedge^{k_1} W \otimes \bigwedge^{k_2} W \otimes \dots)$.

Proof. For $1 \leq i \leq n$ and $1 \leq j \leq m$, let $b_{i,j}$ the box of the $n \times m$ rectangle located at the intersection of the i^{th} row and the j^{th} column and set $x_{i,j} = v_i \otimes w_j$. In the case $d > 0$, we place the Young diagram Y inside the rectangle in a such way that it contains the upper left box $b_{1,1}$. For example:



Let X (respectively X^-) be the subspace of $V \otimes W$ generated by all $x_{i,j}$ with $b_{i,j} \in Y$ (respectively with $b_{i,j} \notin Y$). Note that X is a d-dimensional $U_V \times U_W$-submodule of $V \otimes W$ and X^- is a $(nm - d)$-dimensional $U_V^- \times U_W^-$-submodule of $V \otimes W$. Choose non zero vectors $x \in \bigwedge^d X$ and $x^- \in \bigwedge^{nm-d} X^-$.

We claim that $T^{GL(W)}_{\lambda^T(Y)}(\mathbb{M})$ contains a non-zero U_V-invariant vector of weight $\lambda(Y)$. The vector x is $U_V \times U_W$-invariant of weight $(\lambda(Y), \lambda^T(Y))$. Hence it defines a U_V-invariant element $\overline{x} \in T^{GL(W)}_{\lambda^T(Y)}(\mathbb{M})$. For $y \in \mathbb{M}$, set $\tau(y) = \int y \wedge x^-$, where $\int : \mathbb{M} \to \bigwedge^{mn}(V \otimes W)$ is the projection over the top component of $\bigwedge(V \otimes W)$. Since x^- is U_W^--invariant and $\bigwedge^{nm}(V \otimes W)$ is a trivial U_W^--module, the map $\tau : \mathbb{M} \to \bigwedge^{mn}(V \otimes W)$ is U_W^--equivariant and therefore factors trough $\mathbb{M}_{U_W^-}$. By definition, $x \wedge x^- \neq 0$ therefore $\tau(x) \neq 0$. As the image of x in $\mathbb{M}_{U_W^-}$ is not zero, we have $\overline{x} \neq 0$. Hence $T^{GL(W)}_{\lambda^T(Y)}(\mathbb{M})$ contains a non-zero U_V-invariant vector of weight $\lambda(Y)$, namely $\overline{x}$.

However, by Corollary 1.3 and Theorem 2.1 (ii), the non-zero $GL(V)$-module $T^{GL(W)}_{\lambda^T(Y)}(\mathbb{M})$ is simple. The previous claim and the classification of the simple $GL(V)$-modules by the weight of their U_V-invariant vectors [St] implies that $T^{GL(W)}_{\lambda^T(Y)}(\mathbb{M}) \simeq L_V(\lambda(Y))$. Thus Assertion (i) is proved.

Identify $V \otimes W \simeq W \oplus W \oplus \dots$, where the i^{th}-factor W is $v_i \otimes W$. Accordingly, we get $\mathbb{M} \simeq \bigwedge W \otimes \bigwedge W \dots$ (n times). Thus the eigenspace of weight μ of the $GL(V)$-module $T^{GL(W)}(\mathbb{M})$ is $T^{GL(W)}(\bigwedge^{k_1} W \otimes \bigwedge^{k_2} W \otimes \dots)$, and Assertion (ii) follows. □

Corollary 2.3. *Let Y be a Young diagram of degree d contained in the $n \times m$ rectangle, and let $\mu = \sum_{1\leq k\leq n} k_i\, \epsilon_i^V$ be a polynomial weight of degree d. The dimension of $L_V(\lambda(Y))_\mu$ is the multiplicity (as a direct summand) of the indecomposable $GL(W)$-module $T_W(\lambda^T(Y))$ in $\bigwedge^{k_1} W \otimes \bigwedge^{k_2} W \otimes \dots$.*

Proof. The assertion follows from Corollary 1.3 and Lemma 2.2. □

Remark. By Corollary 2.3, the knowledge of tensor product multiplicities of tilting modules determines the character formula of simple modules. This formula can be compared with Donkin's formula for decomposition numbers. The formula is (see [D2], Lemma 3.1):

$$[T_V(\lambda(Y)) : \nabla_V(\lambda(Y'))] = [\nabla_W(\lambda^T(Y')) : L_W(\lambda^T(Y))].$$

Therefore, each of the following computations (for all $GL(n)$)

(i) all decomposition numbers $[\nabla_V(\mu) : L_V(\lambda)]$,

(ii) the character formula of all the simple modules $L_V(\lambda)$,

(iii) the character formula of all the tilting modules $T_V(\lambda)$,

(iv) the tensor product multiplicities of all the tensor products of two tilting modules,

are equivalent with each other (see [D2] for further details). It should be noted that the determination of the character of all tilting modules is a very difficult problem: e.g. there is no conjecture for them, even for the small group $GL_3(K)$ (in contrast, the character formulas for simple $GL_3(K)$-modules can be obtained very easily). The main observation of the paper is based on the fact that a partial information about tensor product multiplicities (namely, the modular Verlinde formula [GM1], [GM2]) is enough to determine the character formula of a certain class of simple modules.

3. Some multiplicities of tilting $GL(W)$-modules in $\bigwedge(V \otimes W)$

Let W be a vector space of dimension $m < p$. We will use the notations of Section 2 together with the following notations. Set $h_0^W = (\epsilon_1^W)^* - (\epsilon_m^W)^*$, where $((\epsilon_i^W)^*)_{1\leq i \leq m}$ is the dual basis of $Hom(P_W, \mathbb{Z})$ (i.e. h_0^W is the highest coroot of $GL(W)$). Denote by Ω_j^W the set of all weights of the form $\epsilon_{k_1}^W + \epsilon_{k_2}^W + \cdots + \epsilon_{k_j}^W$ with $k_1 < k_2 < \cdots < k_j$. Thus Ω_j^W is the set of weights of $\bigwedge^j W$ and ω_j^W is its highest weight. Set $C_W = \{\lambda \in P_W^+ | \lambda(h_0^W) \leq p - m\}$. Usually, C_W is called the interior of the fundamental alcove.

Lemma 3.1. *Let $\lambda \in P_W^+$. Then p divides $\dim T_W(\lambda)$ if and only if $\lambda \notin C_W$.*

Lemma 3.2. *Let j be an integer with $0 \leq j \leq m$. We have $T_W(\omega_j^W) \simeq \bigwedge^j W$. In particular, the set of weights of $T_W(\omega_j^W)$ is Ω_j^W and each weight appears with multiplicity one.*

Lemma 3.3. *Let $\lambda \in C_W$ and let j with $0 \leq j \leq m$. We have $T_W(\lambda) \otimes T_W(\omega_i^W) = \oplus_\nu T_W(\lambda + \nu)$, where the sum runs over all $\nu \in \Omega_j^W$ such that $(\lambda + \nu) \in P_W^+$.*

References for the previous three lemmas are as follows: Lemma 3.2 follows from the fact that $\bigwedge^j W$ is simple (see [D2] for details). Lemma 3.1 and Lemma 3.3 follow from the main result of [GM1], [GM2] (the modular Verlinde formula). For the peculiar case considered here, there is a quick proof of both lemmas , see Proposition 10 of [M2] and Lemma 12 of [M2] ($m-1$ is the value of $\rho(h_0^W)$ of *loc. cit.*). This quick proof is based on Andersen's linkage principle [A] and on Lemma 1.4..

Lemma 3.4. *Let $k_1, \dots, k_n$ be integers with $0 \le k_i \le m$. We have:*

$$\bigwedge^{k_1} W \otimes \dots \otimes \bigwedge^{k_n} W = T \oplus [\oplus_{(\nu_1,\dots,\nu_n)} T_W(\nu_1 + \dots + \nu_n)],$$

where T is a sum of indecomposable tilting modules of dimension divisible by p and where the sum runs over all n-tuples $(\nu_1, \dots, \nu_n) \in \Omega_{k_1}^W \times \dots \times \Omega_{k_n}^W$ such that $\nu_1 + \nu_2 + \dots + \nu_i$ belongs to C_W, for any i with $1 \le i \le n$.

Proof. Let $\lambda \in P_W^+$ and let k be an integer with $0 \le k \le m$. Assume first that $\lambda \in C_W$. It follows from lemmas 3.1, 3.2 and 3.3 that we have:

$$T_W(\lambda) \otimes \bigwedge^{k} W \simeq T \oplus [\oplus_\nu T_W(\lambda + \nu)], \tag{3.4.1}$$

where T is a sum of indecomposable tilting modules of dimension divisible by p and where the sum runs over all $\nu \in \Omega_k^W$ such that $\lambda + \nu$ belongs to C_W. Next, assume that $\lambda \notin C_W$. From lemmas 1.4 and 3.1, we get:

$$T_W(\lambda) \otimes \bigwedge^{k} W \simeq T, \tag{3.4.2}$$

where T is a sum of indecomposable tilting modules of dimension divisible by p. Note that for $n = 1$, the assertion of Lemma 3.4 is obvious: indeed the conditions $\nu_1 \in C_W$ and $\nu_1 \in \Omega_{k_1}^W$ simply mean $\nu_1 = \omega_{k_1}^W$. Thus, Lemma 3.4 follows, by induction over n, from the assertions (3.4.1) and (3.4.2). □

Example 3.5. For this example, we will consider the case $m = p - 1$. For any $k \in \mathbb{Z}$, set $\theta_k^W = \omega_a^W + b.\omega_m^W$, where $k = a + mb$ and $0 \le a < m$. It is clear that $C_W = \{\theta_k^W | k \in \mathbb{Z}\}$ and Lemma 3.4 can be stated as follows:

$$\bigwedge^{k_1} W \otimes \dots \otimes \bigwedge^{k_n} W = T \oplus T_W(\theta_{k_1 + \dots + k_n}^W),$$

where T is a sum of indecomposable tilting modules of dimension divisible by p. Using Corollary 2.3, we get that for any special weight λ with $m(\lambda) = p - 1$, the module $L_V(\lambda)$ is multiplicity free. Indeed, we recover a well-known fact: for such a weight, we have $\lambda = a\,\omega_i^V + b\,\omega_{i+1}^V$, for some integers a, b, i with $a + b = p - 1$. Set $N = a\,i + b\,(i + 1)$. As $L_V(\lambda)$ is the degree N restricted symmetric power of V (see [Do]), it is multiplicity free.

4. Character of $L_V(\lambda)$, λ being a special weight

We will use the notations of the previous two sections. In particular, the dimensions of V and W are n and m. *We will always assume that $m < p$.* Also denote by C_W^{pol} the set of all polynomial weights in C_W. A polynomial weight $\lambda \in P_V^+$ is called *m-special* if there are integers i, j with $\lambda = \sum_{i \leq k \leq j} a_k \, \omega_k^V$, $\sum_{i \leq k \leq j} a_k \leq m$ and $j - i \leq p - m$. To compare the notion of m-special weights with the notion of special weights given in the introduction, we need the following two observations:

(i) λ is special if and only if $m(\lambda) < p$ and λ is $m(\lambda)$-special,

(ii) if λ is m-special for some $m < p$, then $m(\lambda) \leq m$ and λ is special,

for any dominant polynomial weight λ. In particular, any m-special weight is $m(\lambda)$-special and special. Let $Young(n, m)$ be the set of all m-special Young diagrams of height $\leq n$.

Lemma 4.1. *(i) The map $Y \mapsto \lambda(Y)$ is a bijection from the set $Young(n, m)$ to the set of all m-special weights of P_V^+.*

(ii) The map $Y \mapsto \lambda^T(Y)$ is a bijection from the set $Young(n, m)$ to C_W^{pol}.

Proof of Assertion (i). The map $Y \mapsto \lambda(Y)$ is a bijection between the set of all Young diagrams of height $\leq n$ and the set of all dominant polynomial weights of $GL(V)$. More explicitly, this map is given by: $Y \mapsto \sum_{k \geq 1} \omega^V_{c_k(Y)}$. We have $m(\lambda(Y)) = r_1(Y)$, hence Young diagrams with at most m columns correspond with weights λ with $m(\lambda) \leq m$. Moreover if $r_1(Y) \leq m$, we have $\lambda(Y) = \sum_{i \leq k \leq j} a_k \omega_k^V$, where $i = c_m(Y)$, $j = c_1(Y)$. Thus $\lambda(Y)$ is m-special if and only if Y is m-special.

Proof of Assertion (ii). The map $Y \mapsto \lambda^T(Y)$ is a bijection between the set of all Young diagrams of height $\leq m$ and the set of all dominant polynomial weights of $GL(W)$. We have $\lambda^T(Y) = \sum_{k \geq 1} c_k(Y) \, \epsilon_k^W$, therefore we have $\lambda^T(Y)(h_0^W) = c_1(Y) - c_m(Y)$. Hence Y is m-special, if and only if $\lambda^T(Y)$ belongs to C_W^{pol}. □

Lemma 4.2. *Let $Y \in Young(n, m)$ and let $\mu = \sum_{1 \leq i \leq n} k_i \, \epsilon_i^V$ be a polynomial weight such that $|Y| = |\mu|$. There is a natural bijection Ψ_m from*

(i) the set of all m-semi-standard tableaux of shape Y and weight μ, to

(ii) the set of all n-tuples $(\nu_1, ..., \nu_n) \in \Omega^W_{k_1} \times \cdots \times \Omega^W_{k_n}$ such that $\sum_{1 \leq i \leq n} \nu_i = \lambda^T(Y)$ and such that $\nu_1 + \nu_2 + ... + \nu_i$ belongs to C_W, for any i with $1 \leq i \leq n$.

Proof. Let T be a tableau of shape Y and weight μ. Define n weights $\nu_1, \ldots, \nu_n \in P_W$ by the requirement:

$$\nu_1 + \cdots + \nu_i = \lambda^T(T[i]), \text{ for all } i \text{ with } 1 \leq i \leq n.$$

Note that $T[i] \setminus T[i-1]$ contains exactly k_i boxes. Assume that T is semi-standard. Then any two boxes of $T[i] \setminus T[i-1]$ are located on different columns. Denote by $j_1 < \cdots < j_{k_i}$ the numbering of the non empty columns of $T[i] \setminus T[i-1]$. We have $\nu_i = \epsilon^W_{j_1} + \epsilon^W_{j_2} + \ldots$, hence ν_i belongs to $\Omega^W_{k_i}$. Thus it is clear that the map $\Psi : T \mapsto (\nu_1, ..., \nu_n)$ is a bijection from

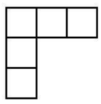
(i) the set of all semi-standard tableaux of shape Y and weight μ, to

(ii) the set of all n-tuples $(\nu_1, ..., \nu_n) \in \Omega^W_{k_1} \times \cdots \times \Omega^W_{k_n}$ such that $\sum_{1 \leq i \leq n} \nu_i = \lambda^T(Y)$ and such that $\nu_1 + \nu_2 + ... + \nu_i$ belongs to P^+_W, for any i with $1 \leq i \leq n$. Indeed, this bijection Ψ is equivalent to the rule of Richardson and Littlewood ([LR], see also [Li]). Denote by Ψ_m the restriction of Ψ to the set of all m-semi-standard tableaux of shape Y and weight μ. As the weights $\nu_1 + \nu_2 + ... + \nu_i$ are polynomial, it follows from Lemma 4.1 (ii) that Ψ_m is the bijection required by Lemma 4.2. $\square$

Theorem 4.3. *Let λ be a special weight. Any weight of $L_V(\lambda)$ is polynomial and for any polynomial weight μ, the dimension of $L_V(\lambda)_\mu$ is the number of $m(\lambda)$-semi-standard tableaux of shape $Y(\lambda)$ and of weight μ.*

Proof. It is well known that the weights of $L_V(\lambda)$ are polynomial ([G]). Set $Y = Y(\lambda)$ and $m = m(\lambda)$ and let $\mu = \sum_{1 \leq i \leq n} k_i \, \epsilon^V_i$ be a polynomial weight. By Corollary 2.3, the dimension of $L_V(\lambda(Y))_\mu$ is the multiplicity of the indecomposable $GL(W)$-module $T_W(\lambda^T(Y))$ in $\bigwedge^{k_1} W \otimes \bigwedge^{k_2} W \otimes \ldots$. By Lemma 4.1 (i), Y belongs to $Young(n, m)$ and by Lemma 4.1 (ii), $\lambda^T(Y)$ belongs to C_W. By Lemma 3.1, the dimension of the tilting $GL(W)$-module $T_W(\lambda^T(Y))$ is not divisible by p. Hence by Lemma 3.4, dim $L_V(\lambda(Y))_\mu$ is the number of all n-tuples $(\nu_1, ..., \nu_n) \in \Omega^W_{k_1} \times \cdots \times \Omega^W_{k_n}$ such that $\sum_{1 \leq i \leq n} \nu_i = \lambda^T(Y)$ and such that $\nu_1 + \nu_2 + ... + \nu_i$ belongs to C_W, for any i with $1 \leq i \leq n$. Hence by Lemma 4.2, dim $L_V(\lambda(Y))_\mu$ is also the number of m-semi-standard tableaux of shape $Y(\lambda)$ and of weight μ. $\square$

Example 4.4. Consider the polynomial dominant weight $\lambda = 2\omega^V_1 + \omega^V_3$ and set $Y = Y(\lambda)$. The Young diagram Y is the hook:



The notion of m-special Young diagrams depends on the characteristic p of the ground field K. In our example, Y is 3-special if and only if $p \geq 5$. Therefore, Theorem 4.3 determines the character formula of the simple $GL(V)$-module $L_V(\lambda)$ for any $p \geq 5$. As the height of Y is 3, we need to require $n \geq 3$, but to find an interesting weight multiplicity, we will assume $n \geq 4$.

Set $\mu = \epsilon^V_1 + \epsilon^V_2 + \epsilon^V_3 + 2\epsilon^V_4$. There are three semi-standard tableaux T, T' and T" of shape Y and weight μ, namely:

$$T : \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad T' : \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad T" : \begin{array}{|c|c|c|}\hline 1 & 4 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

It is clear that for any $p \geq 7$, these three tableaux are 3-semi-standard. Therefore $dim\, L_V(\lambda)_\mu = 3$ if $p \geq 7$ (or if the characteristic is zero). If K is a field of characteristic 5, the semi-standard Young tableau T" is not 3-semi-standard, because T"[3] is not 3-special. Since the other two tabeaux are 3-semi-standard, we have $dim\, L_V(\lambda)_\mu = 2$.

Using only Theorem 4.3, one can get the full character formula of $L_V(\lambda)$ for all $p \geq 5$, but not for $p = 2$ or $p = 3$. However, it is also possible to compute the character formula for $L_V(\lambda)$ in characteric 2 using Theorem 5.3: in such case we get $dim\, L_V(\lambda)_\mu = 1$. Of course, the Young diagram Y is so small that it is also possible to determine the character formula of $L_V(\lambda)$ in all characteristics by an explicit computations, but this is not the goal of the example. Using only theorems 4.3 and 5.3, it is not possible to compute the character formula of $L_V(\lambda)$ in characteristic 3.

5. Proof of the Main Theorem

Say that a polynomial weight $\mu = \sum_{1 \leq i \leq n} k_i\, \epsilon_i^V$ is *reduced* if all k_i are $\leq p-1$. For any polynomial weight μ, the weights $\mu(k)$ occurring in its p-adic decomposition are reduced.

Lemma 5.1. *(i) Any special weight is reduced.*

(ii) Let $\mu = \sum_{k \geq 0} p^k\, \mu_k$ be a polynomial weight where all μ_k are reduced and $\mu_k = 0$ for $k >> 0$. Then $\mu_k = \mu(k)$ for all $k \geq 0$.

(iii) Let λ be a reduced dominant polynomial weight. Then any weight of $L_V(\lambda)$ is reduced.

Proof. Let λ be a special weight. We have $\lambda = \sum_{1 \leq i \leq n} k_i\, \epsilon_i^V$, with $k_1 = m(\lambda) < p$. As λ is dominant, we have $k_i \leq k_1$, and λ is reduced. Thus Assertion (i) holds. Assertion (ii) is obvious. Let λ be a reduced dominant polynomial weight. Let X be the set of all linear combinations $\sum_{1 \leq i \leq n} x_i\, \epsilon_i^V$, where the x_i are real numbers with $0 \leq x_i \leq p-1$. Then $\lambda \in X$ and X is a convex set which is stable by S_n (the Weyl group of $GL_n(K)$). Hence any weight μ of $L_V(\lambda)$ belongs to X, and μ is reduced. Thus Assertion (iii) holds. □

Lemma 5.2. *Let λ be a polynomial dominant weight of the form $\lambda = \sum_{k \geq 0} p^k\, \lambda_k$, where are all λ_k are reduced and dominant. Let μ be a polynomial weight. Then all weights $\lambda(k)$ are dominant and we have:*

$$L_V(\lambda)_\mu \simeq L_V(\lambda(0))_{\mu(0)} \otimes L_V(\lambda(1))_{\mu(1)} \otimes \dots.$$

Proof. The weights $\lambda(k)$ are dominant, because $\lambda(k) = \lambda_k$ (Lemma 5.1 (ii)). We only stated this obvious fact to explain the notation $L_V(\lambda(k))$. Moreover the infinite tensor product is indeed finite, because $\lambda(k) = \mu(k) = 0$ and $L_V(\lambda(k))_{\mu(k)} = K$ for $k >> 0$.

For $g = (g_{i,j})_{1 \leq i,j \leq n} \in GL_n(K)$, set $Fr(g) = (g_{i,j}^p)_{1 \leq i,j \leq n}$. The map $Fr : GL_n(K) \to GL_n(K)$, called the Frobenius map, is a morphism of groups. Note that any reduced dominant polynomial weight is restricted (as it is defined by Steinberg [St]). Therefore, by Steinberg's product formula (see [St], Theorem 41), there is an isomorphism $L(\lambda) \simeq L(\lambda(0)) \otimes L(\lambda(1)) \otimes \dots$, where the action of $GL_n(K)$ on the k^{th}-factor is shifted by Fr^k. Thus we have $L(\lambda)_\mu = \oplus_{(\mu_0,\mu_1,\dots)} L(\lambda(0))_{\mu_0} \otimes L(\lambda(1))_{\mu_1} \otimes \dots$, where the sum runs over all tuples $(\mu_k)_{k \geq 0}$ such that $\mu = \sum_{k \geq 0} p^k \mu_k$ and each μ_k is a weight of $L(\lambda(k))$. By Assertion (iii) of Lemma 5.1, the weights μ_k are reduced. Then, by Assertion (ii) of Lemma 5.1, we have $\mu_k = \mu(k)$. Thus Lemma 5.2 holds. □

In the introduction, we have already noticed that $\mathcal{C}_n$ and the $GL_n(K)$-module $L_V(\lambda)$ (λ being a polynomial and dominant weight) are well defined also for $n = \infty$.

Theorem 5.3. *Let $\lambda \in \mathcal{C}_n$ where n is finite or infinite. Any weight of the $GL_n(K)$-module $L_V(\lambda)$ is polynomial, and for any polynomial weight μ, we have* $\dim L_V(\lambda)_\mu = \prod_{k \geq 0} N(\lambda(k), \mu(k))$.

Proof. First assume that n is finite. By Assertion (i) of Lemma 5.1, any special weight is reduced. Hence Theorem 5.3 follows from Lemma 5.2 and Theorem 4.3. The case n infinite follows by inductive limit. □

6. Semi-simplicity of restrictions to Young subgroups

Let us consider $GL_{n-1}(K)$ as the subgroup of $GL_n(K)$ as usual. For any $GL_{n-1}(K)$-module L, denotes by $ch(L)$ its character. Set $V' = K^{n-1}$. Therefore the simple $GL_{n-1}(K)$-modules will be denoted by $L_{V'}(\lambda')$, with $\lambda' \in P_{V'}^+$.

Lemma 6.1. *Let $\lambda \in P_V^+$ and let A be a finite subset of $P_{V'}^+$. Assume that $ch(L_V(\lambda)|_{GL_{n-1}(K)}) = \sum_{\lambda' \in A} ch(L_{V'}(\lambda'))$. Then we have $L_V(\lambda)|_{GL_{n-1}(K)} = \oplus_{\lambda' \in A} L_{V'}(\lambda')$. In particular $L_V(\lambda)|_{GL_{n-1}(K)}$ is semi-simple.*

Proof. As the characters of simple $GL_{n-1}(K)$-modules are linearly independent, the module $L_V(\lambda)|_{GL_{n-1}(K)}$ has a composition series in which each $L_{V'}(\lambda')$, for all $\lambda' \in A$, occurs exactly once. Note that $L_V(\lambda)$ carries a non degenerate contravariant form. Let S be a simple $GL_{n-1}(K)$-submodule of $L_V(\lambda)|_{GL_{n-1}(K)}$.

Thus $L_V(\lambda)|_{GL_{n-1}(K)}/S^\perp$ is isomorphic to S. As S does not occur as a quotient of $L_V(\lambda)|_{GL_{n-1}(K)}/S$, we have $S \cap S^\perp = 0$. Hence S is a direct summand and $L_V(\lambda)|_{GL_{n-1}(K)}$ is semi-simple. Thus, we have $L_V(\lambda)|_{GL_{n-1}(K)} = \oplus_{\lambda' \in A} L_{V'}(\lambda')$. $\square$

Let $m < p$. For any Young diagram $Y \in Young(n, m)$, let $V(Y)$ be the set of all Young diagrams Y' such that:
(i) $c_k(Y') \leq c_k(Y) \leq c_k(Y') + 1$, for all $k \geq 1$,
(ii) $Y' \in Young(n-1, m)$.
These conditions are indeed equivalent to the fact that there is a m-semi-standard tableau T of shape Y such that $T[n-1] = Y'$, $T[n] = Y$. Also the definition of $V(Y)$ is independent of m. Namely if Y is m-special and also m'-special for some $m' \neq m$, then condition (ii) is automatically satisfied.

Theorem 6.2. *For $\lambda \in \mathcal{C}_n$, set $Y_k = Y(\lambda(k))$ for all $k \geq 0$. Then we have:*

$$L_V(\lambda)|_{GL_{n-1}(K)} = \oplus_{(Y_0', Y_1', \dots)} L_{V'}(\textstyle\sum_{k \geq 0} p^k \lambda(Y_k')),$$

where the direct sum runs over all tuples $(Y_0', Y_1', \dots) \in V(Y_0) \times V(Y_1) \times \dots$. In particular, $L_V(\lambda)|_{GL_{n-1}(K)}$ is semi-simple. Moreover each simple direct summand occurs with multiplicity one, and its highest weight is in $\mathcal{C}_{n-1}$.

Proof. Let $Y \in Young(n, m)$. There is a natural bijection from
(i) the set of all m-semi-standard tableaux T of shape Y with labels $\leq n$, to
(ii) the set of all pairs (Y', T'), where $Y' \in V(Y)$ and T' is a m-semi-standard tableaux of shape Y' with labels $\leq n-1$.
Explicitly, $Y' = T[n-1]$ and T' is the tableau induced by T. It follows from Theorem 5.3 that we have:

$$ch(L_V(\lambda)|_{GL_{n-1}(K)}) = \textstyle\sum_{(Y_0', Y_1', \dots)} ch(L_{V'}(\sum_{k \geq 0} p^k \lambda(Y_k'))),$$

where the sum runs over all tuples $(Y_0', Y_1', \dots) \in V(Y_0) \times V(Y_1) \times \dots$. Let $(Y_0', Y_1', \dots)$ be such a tuple and set $\lambda' = \sum_{k \geq 0} p^k \lambda(Y_k')$. It follows from the assertions (i) and (ii) of Lemma 5.1 that $(\lambda(Y_k'))_{k \geq 0}$ are the terms of the p-adic expansion of the polynomial weight λ'. Hence the decomposition $\lambda' = \sum_{k \geq 0} (p^k \lambda(Y_k'))$ with $(Y_0', Y_1', \dots) \in V(Y_0) \times V(Y_1) \times \dots$ is unique. Thus $L_{V'}(\lambda')$ occurs exactly once in the composition series of $L(\lambda)|_{GL_{n-1}(K)}$ and the theorem follows from Lemma 6.1. $\square$

Lemma 6.3. *Let $G_1 \dots, G_k$ be k groups and let L be a finite dimensional $G_1 \times \dots \times G_k$-module. Assume that L is semi-simple as G_i-module for all $1 \leq i \leq k$. Then L is semi-simple as $G_1 \times \dots \times G_k$-module.*

A proof of this lemma can be found in [K] (lemma 1.6). $\square$

For any k-tuple $(a_1, \dots a_k)$ of non-negative integers with $n = a_1 + a_2 + \dots$, there is a natural embedding of $GL_{a_1}(K) \times \cdots \times GL_{a_k}(K)$ inside $GL_n(K)$.

Corollary 6.4. *Let $\lambda \in \mathcal{C}_n$. As a $GL_{a_1}(K) \times \cdots \times GL_{a_k}(K)$-module, $L_V(\lambda)$ is semi-simple.*

Proof. Using Theorem 6.2, we prove by induction over b that $L_V(\lambda)|_{GL_{n-b}(K)}$ is semi-simple for all $b \leq n$. Thus the corollary follows from Lemma 6.3. $\square$

Remark. Let λ be a special weight of degree n. The weight space $L_V(\lambda)_{\epsilon_1^V + \cdots + \epsilon_n^V}$ is the simple representation of the symmetric group S_n associated with the Young diagram transposed of $Y(\lambda)$ (see [J]). Its dimension is computed by Theorem 4.3, and its restriction to the subgroup $S_{a_1} \times \cdots \times S_{a_k}$ is semi-simple by Corollary 6.4. These two results for the symmetric groups were already established: we recover respectively the main results of [M2] (dimension formula) and of [K] (semi-simplicity). Indeed the proof of the semi-simplicity of $L_V(\lambda)|_{GL_{a_1}(K)\times\cdots\times GL_{a_k}(K)}$ is similar to the proof of [K].

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