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The Łojasiewicz exponent of an analytic function at an isolated zero

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Abstract. Let f be a real analytic function defined in a neighborhood of $0 \in \mathbb{R}^n$ such that $f^{-1}(0) = \{0\}$. We describe the smallest possible exponents α, β, θ for which we have the following estimates: $|f(x)| \geq c|x|^\alpha$, $|\operatorname{grad} f(x)| \geq c|x|^\beta$, $|\operatorname{grad} f(x)| \geq c|f(x)|^\theta$ for x near zero with $c > 0$. We prove that $\alpha = \beta + 1$, $\theta = \beta/\alpha$. Moreover $\beta = N + a/b$ where $0 \leq a < b \leq N^{n-1}$. If f is a polynomial then $|f(x)| \geq c|x|^{(\deg f-1)^n+1}$ in a small neighborhood of zero.

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1. Results

Let $f : U \to \mathbb{R}$ be an analytic function defined in a neighborhood U of $0 \in \mathbb{R}^n$. Assume that f has an *isolated zero* at the origin i.e. $f^{-1}(0) \cap W = \{0\}$ for some neighborhood W of zero. Then also $\operatorname{grad} f(x)$ is nonzero for x close to the origin. One of the consequences of the classical Łojasiewicz inequality (see [BM, Theorem 6.4]) is that there exist constants $c, R > 0$ and exponents α, β, θ such that $|f(x)| \geq c|x|^\alpha$, $|\operatorname{grad} f(x)| \geq c|x|^\beta$, $|\operatorname{grad} f(x)| \geq c|f(x)|^\theta$ for all $|x| \leq R$. The aim of this article is a description of the smallest possible exponents for which the above estimates hold true.

Definition 1.1. *By the Łojasiewicz exponent $\ell_0(f, g)$ for the inequality $|f(x)| \geq c|g(x)|^\alpha$ we mean the number*

$$\inf\{\, \alpha \in \mathbb{R}_+ : \exists c, R > 0 \quad |f(x)| \geq c|g(x)|^\alpha \quad \forall |x| \leq R \,\}.$$

Definition 1.2. *Let $f : U \to \mathbb{R}$ be an analytic function defined in an open set $U \subset \mathbb{R}^n$. By the polar curve Γ_v in the direction $v \in \mathbb{R}^n \setminus \{0\}$ we mean the set $\Gamma_v = (\operatorname{grad} f)^{-1}(\mathbb{R}v)$.*

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If $h : (-\delta, \delta) \to \mathbb{R}^n$ is a nonzero analytic mapping then by definition, the order (at zero) of h, denoted by $\nu(h)$, is the largest integer k such that $t^{-k}h(t)$ is bounded near zero. This definition agrees with the classical one for analytic functions and as we show later can be naturally extended to continuous subanalytic maps.

Our main result is

Theorem 1.3. *Let $f : U \to \mathbb{R}$ be an analytic function defined in some neighborhood U of $0 \in \mathbb{R}^n$. Assume that $f^{-1}(0) = \{0\}$. Then there exists a proper linear subspace $L \subset \mathbb{R}^n$ such that for every $v \in \mathbb{R}^n \setminus L$ there is an analytic curve $\gamma : (-1, 1) \to \Gamma_v$, $\gamma(0) = 0$ for which:*

(i) the Łojasiewicz exponent α_0 for inequality $|f(x)| \geq C|x|^\alpha$ is equal $\alpha_0 = \nu(f \circ \gamma)/\nu(\gamma)$,

(ii) the Łojasiewicz exponent β_0 for inequality $|\operatorname{grad} f(x)| \geq C|x|^\beta$ is equal $\beta_0 = \nu(\operatorname{grad} f \circ \gamma)/\nu(\gamma)$,

(iii) the Łojasiewicz exponent θ_0 for inequality $|\operatorname{grad} f(x)| \geq C|f(x)|^\theta$ is equal $\theta_0 = \nu(\operatorname{grad} f \circ \gamma)/\nu(f \circ \gamma)$.

Moreover $\beta_0 = \alpha_0 - 1$, $\theta_0 = \beta_0/\alpha_0$.

The above theorem says that Łojasiewicz exponents α_0, β_0, θ_0 can be computed using parametrizations of "generic" polar curves. Every polar curve Γ_v such that $v \in \mathbb{R}^n \setminus L$ is good from this point of view. In particular at least one of curves $\Gamma_{e^1}, \ldots, \Gamma_{e^n}$ (where $e^1, \ldots, e^n$ is a standard basis of $\mathbb{R}^n$) is good. However the theorem does not say which one of them.

Example. $f(x) = x_1^4 + x_2^2 + x_3^2$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Set $L = \{0\} \times \mathbb{R}^2 \subset \mathbb{R}^3$. For every $v \in L$, $v \neq 0$ the polar curve $\Gamma_v = \{\, x \in \mathbb{R}^3 : \exists \lambda \in \mathbb{R} \quad (4x_1^3, 2x_2, 2x_3) = \lambda(0, v_2, v_3) \,\}$ is the straight line in the direction v. Taking the parametrization $\gamma : (-1, 1) \to \Gamma_v$, $\gamma(t) = tv$ we get $f(\gamma(t)) = |tv|^2$, $|\gamma(t)| = |tv|$. Hence $\nu(f \circ \gamma)/\nu(\gamma) = 2$.

One can show directly that the Łojasiewicz exponent for the inequality $|f(x)| \geq c|x|^\alpha$ equals 4. Therefore all polar curves Γ_v where $v \in L$ are bad from the point of view of Theorem 1.3. This example shows that this theorem cannot be improved by replacing the linear subspace L by a smaller set $L' \subset L$.

The idea of using polar curves to compute Łojasiewicz exponents comes from Teissier [Te]. He has shown a counterpart of Theorem 1.3 in the complex case. If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a holomorphic function with an isolated singularity at zero then a "generic complex polar curve" has a parametrization such that the analogue of parts (ii) and (iii) of Theorem 1.3 hold. There is also a formula $\theta_0 = \beta_0/(\beta_0 + 1)$ for Łojasiewicz exponents in complex case.

One may ask — can a version of Theorem 1.3 be formulated for real analytic functions with an isolated singularity at 0? The following example due to Kuo shows that we should not expect it.

Example. Let $f(x,y) = x^3 + 3xy^4$. The function f has an isolated singularity at 0. However for all polar curves but one the origin is not an accumulation point of Γ_v. Moreover the Łojasiewicz exponents for inequalities $|\operatorname{grad} f(x)| \geq c|x|^\beta$, $|\operatorname{grad} f(x)| \geq c|f(x)|^\theta$ are $\beta_0 = 4$ and $\theta_0 = 2/3$ respectively so $\theta_0 \neq \beta_0/(\beta_0+1)$.

The second result of this paper is

Theorem 1.4. *Under assumptions and notations of Theorem 1.3, $\beta_0 = N + a/b$ where a, b, N are integers such that $0 \leq a < b \leq N^{n-1}$.*

Let us denote L_n the set of the Łojasiewicz exponents for inequalities $|f(x)| \geq c|x|^\alpha$ where $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are analytic functions with an isolated zero. It is easily seen that L_1 is the set of positive integers $\{1, 2, 3, \dots\}$. The author showed in [Gw] using Puiseux expansions that $L_2 = 2L_1 \cup 2\{\, N + a/b : 0 < a < b < N\,\} = \{\, 2, 4, 6, 7, 8, 8\frac{2}{3}, \dots\}$. The question, how large are sets L_n for $n \geq 3$ remains open.

The last result measeures the growth of polynomial functions.

Theorem 1.5. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function with an isolated zero at the origin. Then*

$$|F(x)| \geq \operatorname{const} |x|^{(\deg F - 1)^n + 1}$$

in a small neighborhood of zero.

2. Proofs

First we extend the definition of order to continuous subanalytic functions. Let $g : [0, \epsilon) \to \mathbb{R}$ be a continuous subanalytic function. Here and subsequently we assume that $g \neq 0$ in every neighborhood of zero. Then there exist (see [BoR, Lemma 3]) a nonnegative rational number ν and a continuous function $g_1 : [0, \delta] \to \mathbb{R}$ $(0 < \delta < \epsilon)$ such that for all $t \in [0, \delta]$ $g_1(t) \neq 0$ and $g(t) = t^\nu g_1(t)$.

It is obvious that the exponent ν is uniquely determined by the function g (even by a germ of g at zero). We call this number the order (at zero) of g and will denote it by $\nu(g)$. We extend the notion of order to subanalytic continuous maps putting $\nu(\phi) = \nu(|\phi|)$ for $\phi : [0, \epsilon) \to \mathbb{R}^n$.

Property 2.1. *Let $g, h : [0, \epsilon) \to \mathbb{R}$ be continuous subanalytic functions non-vanishing in every neighborhood of zero and let r be a positive rational number. Then:*

(i) $\nu(g^r) = r\nu(g)$, $\nu(gh) = \nu(g) + \nu(h)$,

(ii) $\nu(g) \leq \nu(h)$ *if and only if there exist $c, \delta > 0$ such that $|g(t)| \geq c|h(t)|$ for all $t \in [0, \delta]$.*

Proof. The proof of (i) is straightforward. Therefore we only prove (ii). According to definition of order there exist $\delta > 0$ and continuous functions g_1, h_1 such that for all $t \in [0, \delta]$ $\quad h(t) = t^{\nu(h)} h_1(t)$, $g(t) = t^{\nu(g)} g_1(t)$, $h_1(t) \neq 0$, $g_1(t) \neq 0$. If $\nu(g) \leq \nu(h)$ then $|g(t)| \geq c|h(t)|$ for $t \in [0, \min\{1, \delta\}]$, where $c = \inf_{0 \leq t \leq \delta} |g_1(t)/h_1(t)|$. Conversely, if $|g(t)| \geq c|h(t)|$ for $t \in [0, \delta]$ then $t^{\nu(g)-\nu(h)} \geq c|h_1(t)/g_1(t)|$ in the interval $(0, \delta)$. From this inequality it follows that $\nu(g) - \nu(h) \leq 0$. Hence $\nu(g) \leq \nu(h)$. $\square$

Let us recall the classical curve–selection lemma (see [Hi, page 482]).

Lemma 2.2. *Let $A \subset \mathbb{R}^n$ be a subanalytic set. If $0 \in \mathrm{cl}(A)$ then there exists an analytic curve $\gamma : (-1, 1) \to \mathbb{R}^n$ such that $\gamma(0) = 0$ and $\gamma((0, 1)) \subset A$.*

In the following lemma we reformulate the main result of [BoR] in the case of functions with isolated zeros. Let K denote a closed ball $\{x \in \mathbb{R}^n : |x| \leq r\}$.

Lemma 2.3. *Let $f, g : K \to [0, \infty)$ be continuous subanalytic functions such that $f^{-1}(0) = g^{-1}(0) = \{0\}$ and let*

$$K^* = \{\, x \in K : \forall y \in K \quad g(y) = g(x) \Rightarrow f(y) \geq f(x) \,\}.$$

Then

(i) K^* is a subanalytic set, $0 \in \mathrm{cl}(K^* \setminus \{0\})$

(ii) if $\gamma : (-1, 1) \to \mathbb{R}^n$ is an analytic curve such that $\gamma(0) = 0$, $\gamma((0, 1)) \subset K^* \setminus \{0\}$, then $\ell_0(f, g) = \nu(f \circ \gamma)/\nu(g \circ \gamma)$.

Proof. Part (i) of the lemma is proved with all details in [BoR]. Here we present only the sketch of the proof. All properties of subanalytic sets which we use, can be found in [BM].

Let $A = \{(x, y) \in K \times K : g(x) = g(y)\}$, $B = \{(x, y) \in K \times K : f(x) > f(y)\}$. These are subanalytic sets. The set K^* equals $K \setminus \pi(A \cap B)$ where $\pi(x, y) = x$ is a projection. The intersection and the complement of subanalytic sets are subanalytic. Furthermore a projection maps relatively compact subanalytic sets onto subanalytic sets. Therefore K^* is subanalytic.

To show that 0 is an accumulation point of K^* it is enough to check that every ball $K_\epsilon = \{x \in \mathbb{R}^n : |x| < \epsilon\}$ $(0 < \epsilon < r)$ has a non–empty intersection with $K^* \setminus \{0\}$. Set $m = \inf\{g(y) : y \in K \setminus K_\epsilon\}$ and consider the level set $L = g^{-1}(m/2)$. Since L is compact, there exists $x \in L$ such that $f(x) \leq f(y)$ for all $y \in L$. Clearly $x \in K^* \cap K_\epsilon$ and $x \neq 0$.

Proof of (ii). Let γ be an analytic curve from the statement of the lemma. Set $\alpha = \nu(f \circ \gamma)/\nu(g \circ \gamma)$. By Property 2.1 (i) we have $\nu(f \circ \gamma) = \nu((g \circ \gamma)^\alpha)$. Thus by 2.1 (ii) there are positive constants c, δ such that

$$f(\gamma(t)) \geq c g(\gamma(t))^\alpha \qquad \text{for } t \in [0, \delta]. \tag{1}$$

Since g is continuous, there exists $R > 0$ such that for all $|x| \leq R$ $g(x) \leq g(\gamma(\delta))$. Fix $x \in K$ with $|x| \leq R$. By continuity of $g \circ \gamma$, there exists $t \in [0, \delta]$ such that

$g(\gamma(t)) = g(x)$. By the definition of K^* and by (1) we get $f(x) \geq f(\gamma(t)) \geq cg(\gamma(t))^\alpha = cg(x)^\alpha$. Therefore

$$f(x) \geq cg(x)^\alpha \qquad \text{for } |x| \leq R. \tag{2}$$

To end the proof it is enough to show that $\alpha = \nu(f \circ \gamma)/\nu(g \circ \gamma)$ is the smallest possible exponent in the Łojasiewicz inequality. It follows from the following claim applied to the curve γ.

Claim 1. Let $\phi : (-1, 1) \to K$ be an analytic curve such that $\phi(0) = 0$, $\phi \neq 0$. If $f(x) \geq cg(x)^\beta$ in some neighborhood of zero then $\beta \geq \nu(f \circ \phi)/\nu(g \circ \phi)$.

Proof of the claim. Under assumptions of Claim 1 there exists $\tau > 0$ such that $f(\phi(t)) \geq cg(\phi(t))^\beta$ for $t \in [0, \tau]$. By Property 2.1(ii) we have $\nu(f \circ \phi) \leq \nu((g \circ \phi)^\beta)$. Hence by 2.1 (i) $\beta \geq \nu(f \circ \phi)/\nu(g \circ \phi)$. □

Under assumptions of Lemma 2.3 we have

Corollary 2.4. *The Łojasiewicz exponent $\ell_0(f, g)$ is a positive rational number. There exists a positive constant C such that $|f(x)| \geq C|g(x)|^{\ell_0(f,g)}$ in a neighborhood of zero. Furthermore:*
- *(i) for every analytic curve $\phi : (-1, 1) \to \mathbb{R}^n$ such that $\phi(0) = 0$, $\phi \neq 0$ we have $\ell_0(f, g) \geq \nu(f \circ \phi)/\nu(g \circ \phi)$,*
- *(ii) there exists an analytic curve $\gamma : (-1, 1) \to \mathbb{R}^n$, $\gamma(0) = 0$, $\gamma \neq 0$ such that $\ell_0(f, g) = \nu(f \circ \gamma)/\nu(g \circ \gamma)$.*

By the curve–selection lemma and part (i) of Lemma 2.3 there exists an analytic curve γ satisfying assumptions of part (ii) of Lemma 2.3. This proves (ii). The rest of Corollary 2.4 follows from inequality (2) and from Claim 1.

Proof of Theorem 1.3. The one–dimensional case, being simple, is left to the reader. Further we will assume that the function f is defined in a neighborhood U of $0 \in \mathbb{R}^n$ where $n \geq 2$. Consider a ball $K = \{ x \in \mathbb{R}^n : |x| \leq \epsilon \}$ contained in U. Since $f(x) \neq 0$ for $x \in K \setminus \{0\}$ and $K \setminus \{0\}$ is connected, f restricted to $K \setminus \{0\}$ has a constant sign. Without loss of generality we may assume that $f(x) > 0$ for all $x \in K \setminus \{0\}$ (otherwise we replace f by $-f$).

Let

$$A = \{ x \in K : \forall y \in K \quad |y| = |x| \Rightarrow f(y) \geq f(x) \}.$$

Consider the tangent cone $C(A)$ defined by the following condition:
$a \in C(A)$ *if and only if there exist sequences* $x_i \in A$ *and* $\lambda_i \in \mathbb{R}$ *such that* $\lim_{i\to\infty} x_i = 0$ *and* $\lim_{i\to\infty} \lambda_i x_i = a$.

We will check that there exists $a \in C(A)$ such that $a \neq 0$. Take any sequence $x_i \in A \setminus \{0\}$ converging to zero. Then from the sequence of points $(1/|x_i|)x_i$ lying on the unit sphere one can choose a subsequence convergent to some a, $|a| = 1$. Clearly $a \in C(A)$.

The linear subspace L appearing in the statement of the theorem is defined as follows
$$L = \{ y \in \mathbb{R}^n : \forall u \in C(A) \quad \langle y, u \rangle = 0 \} \tag{3}$$

Fix $v \in \mathbb{R}^n \setminus L$. By the definition of L there exists $u \in C(A)$ such that $\langle v, u \rangle \neq 0$. Choose a constant $c > 0$ such that $|\langle v, u \rangle| > c|v||u|$ (e.g. we can take $c = |\langle v, u \rangle|/(2|v||u|)$.

Let us define an open cone C

$$C = \{ x \in \mathbb{R}^n : |\langle v, x \rangle| > c|v||x| \}. \tag{4}$$

Claim 1. $A \cap C$ is a subanalytic set, $0 \in \mathrm{cl}(A \cap C)$.

Proof of Claim 1. For $u \in C(A)$ as above we have $u \in C$. Let $x_i \in A$ and $\lambda_i \in \mathbb{R}$ be sequences such that $\lim_{i \to \infty} \lambda_i x_i = u$ and $\lim_{i \to \infty} x_i = 0$. Since C is open, $\lambda_i x_i \in C$ for i large enough. Hence $x_i \in C$ for sufficiently large i. This proves that $0 \in \mathrm{cl}(A \cap C)$.

By Lemma 2.3 the set A is subanalytic. The cone C is also subanalytic (C is even semialgebraic). Thus $A \cap C$ as an intersection of subanalytic sets is subanalytic. The claim follows.

Let us define the new norm in $\mathbb{R}^n$ by a formula

$$\|x\| = \max\{|x|, |\langle v, x \rangle|/c|v|\} \tag{5}$$

One checks easily that $\|x\| > |x|$ for $x \in C$ and $\|x\| = |x|$ otherwise.

Claim 2. $\ell_0(f, |\ |) = \ell_0(f, \|\ \|)$

The claim follows from inequalities $\|x\| \geq |x| \geq c\|x\|$ and from the definition of the Łojasiewicz exponent.

Consider the following set

$$B = \{ x \in K : \forall y \in K \quad \|y\| = \|x\| \Rightarrow f(y) \geq f(x) \}.$$

Claim 3. $B \setminus \{0\} \subset \Gamma_v \cap C$ in some neighborhood of zero.

Proof of Claim 3. This is the key point of the proof of Theorem 1.3. By the curve–selection lemma and Claim 1 there exists an analytic curve $\phi : (-1, 1) \to \mathbb{R}^n$, $\phi(0) = 0$ such that $\phi((0, 1)) \subset A \cap C$. Since a function $f \circ \phi$ is real analytic, its derivative has a finite number of zeros in a small neighborhood of zero. Thus for some $0 < \delta < 1$ $f \circ \phi$ is strictly increasing in the interval $[0, \delta]$.

Set $R = |\phi(\delta)|$ and consider arbitrary $y \in B \setminus \{0\}$ such that $\|y\| < R$. We shall check that $y \in \Gamma_v \cap C$.

By continuity of $|\phi|$ there exists t_1 $(0 < t_1 < \delta)$ such that $|\phi(t_1)| = \|y\|$. The point $x_1 = \phi(t_1)$ belongs to the cone C. Hence $|x_1| < \|x_1\|$.

By continuity of $\|\phi\|$ there is t_2 $(0 < t_2 < t_1)$ such that $\|\phi(t_2)\| = |x_1|$. Put $x_2 = \phi(t_2)$. Since $f \circ \phi$ increases in the interval $[0, \delta]$, we conclude that $f(x_2) < f(x_1)$. We have also $f(y) \leq f(x_2)$ because $y \in B$ and $\|x_2\| = \|y\|$. Therefore $f(y) < f(x_1)$. Since $x_1 \in A$, this inequality implies that $|y| \neq |x_1| = \|y\|$. Both norms of y do not coincide. Thus $y \in C$.

Put $r = \|y\|$. For every $x \in C$ such that $\langle v, x\rangle = \pm rc|v|$ we have $\|x\| = \|y\|$ and consequently $f(x) \geq f(y)$, since $y \in B$. We see that y is the solution of the following problem: find $x \in C$ satisfying a condition $\langle v, x\rangle = \pm rc|v|$ with the smallest value of $f(x)$. By the method of Lagrange's multipliers there is a constant λ such that $\operatorname{grad} f(y) = \lambda v$. Therefore $y \in \Gamma_v$. The claim follows.

Proof of (i). By the curve–selection lemma and Lemma 2.3 there exists an analytic curve $\psi : (-1, 1) \to \mathbb{R}^n$, $\psi(0) = 0$, $\psi((0, 1)) \subset B \setminus \{0\}$ such that $\ell_0(f, \|\ \|) = \nu(f \circ \psi)/\nu(\|\psi\|)$. Furthermore by Claim 3 we get $\psi((0, \tau)) \subset \Gamma_v \cap C$ for some τ, $0 < \tau \leq 1$. Since Γ_v is an analytic set, there exist τ_1, $0 < \tau_1 < \tau$ such that $\psi((-\tau_1, \tau_1)) \subset \Gamma_v$. Set $\gamma(t) = \psi(\tau_1 t)$. We obtained an analytic curve γ such that $\gamma(0) = 0$, $\gamma((-1, 1)) \subset \Gamma_v$, and $\gamma((0, 1)) \subset \Gamma_v \cap C$. From Claim 2 it follows that $\alpha_0 = \ell_0(f, |\ |) = \ell_0(f, \|\ \|) = \nu(f \circ \gamma)/\nu(\|\gamma\|) = \nu(f \circ \gamma)/\nu(\gamma)$. This ends the proof of (i).

To finish the proof we shall use two claims.

Claim 4. The function $|\operatorname{grad} f|$ has an isolated zero at the origin.

Claim 5. For any analytic curve $\phi : (-1, 1) \to U$, $\phi(0) = 0$, $\phi \neq 0$, we have $\nu(f \circ \phi) \geq \nu(\operatorname{grad} f \circ \phi) + \nu(\phi)$. For the curve γ we have
$\nu(f \circ \gamma) = \nu(\operatorname{grad} f \circ \gamma) + \nu(\gamma)$.

We prove these claims later.

Proof of (ii). By Claim 4 we may assume (shrinking the ball K if necessary) that $\operatorname{grad} f(x) \neq 0$ for all $x \in K \setminus \{0\}$. Thus, by Corollary 2.4, there exists an analytic curve $\phi : (-1, 1) \to K$, $\phi(0) = 0$, $\phi \neq 0$ for which $\beta_0 = \nu(\operatorname{grad} f \circ \phi)/\nu(\phi)$. Using Claim 5 and Corollary 2.4 again we get

$$\beta_0 = \nu(\operatorname{grad} f \circ \phi)/\nu(\phi) \leq \nu(f \circ \phi)/\nu(\phi) - 1 \leq \alpha_0 - 1.$$

For the curve γ we have

$$\alpha_0 = \nu(f \circ \gamma)/\nu(\gamma) = \nu(\operatorname{grad} f \circ \gamma)/\nu(\gamma) + 1 \leq \beta_0 + 1.$$

From these inequalities we get $\beta_0 = \nu(\operatorname{grad} f \circ \gamma)/\nu(\gamma) = \alpha_0 - 1$.
Proof of (iii). By Corollary 2.4 there exists an analytic curve $\psi : (-1, 1) \to K$, $\psi(0) = 0$, $\psi \neq 0$ for which $\theta_0 = \nu(\operatorname{grad} f \circ \psi)/\nu(f \circ \psi)$. By Claim 5 we have

$$\theta_0 = \nu(\operatorname{grad} f \circ \psi)/\nu(f \circ \psi) \leq 1 - \nu(\psi)/\nu(f \circ \psi) \leq 1 - 1/\alpha_0.$$

For the curve γ we have

$$\theta_0 \geq \nu(\operatorname{grad} f \circ \gamma)/\nu(f \circ \gamma) = 1 - \nu(\gamma)/\nu(f \circ \gamma) = 1 - 1/\alpha_0.$$

Collecting together these inequalities we obtain $\theta_0 = \nu(\operatorname{grad} f \circ \gamma)/\nu(f \circ \gamma) = 1 - 1/\alpha_0 = \beta_0/\alpha_0$.

It remains to prove claims 4 and 5.

Proof of Claim 4. Suppose to the contrary that $0 \in \operatorname{cl}((\operatorname{grad} f)^{-1}(0) \backslash \{0\})$. Then by the curve–selection lemma there exists an analytic curve $\phi : (-1, 1) \to \mathbb{R}^n$, $\phi(0) = 0$, $\phi \neq 0$ such that $\operatorname{grad} f(\phi(t)) = 0$ for $t \in (0, 1)$. Since the derivative $(f \circ \phi)' = \langle \operatorname{grad} f \circ \phi, \phi' \rangle$ vanishes in the interval $(0, 1)$, $f(\phi(t)) = 0$ for $t \in [0, 1)$. Therefore $0 \in \operatorname{cl}(f^{-1}(0) \setminus \{0\})$ – a contradiction.

Proof of Claim 5. For any analytic function h of positive order, we have $\nu(h) = \nu(h') + 1$. Hence $\nu(f \circ \phi) - \nu(\operatorname{grad} f \circ \phi) - \nu(\phi) = \nu((f \circ \phi)') + 1 - \nu(\operatorname{grad} f \circ \phi) - (\nu(\phi') + 1) = \nu\langle \operatorname{grad} f \circ \phi, \phi' \rangle - \nu(\operatorname{grad} f \circ \phi) - \nu(\phi')$. Therefore it suffices to prove inequality

$$\nu\langle \operatorname{grad} f \circ \phi, \phi' \rangle \geq \nu(\operatorname{grad} f \circ \phi) + \nu(\phi') \tag{6}$$

and show that when we replace ϕ by the curve γ we get equality. The above inequality is a consequence of the estimate $|\langle \operatorname{grad} f \circ \phi, \phi' \rangle| \leq |\operatorname{grad} f \circ \phi||\phi'|$ and Property 2.1.

For the curve γ we have $c|v||\gamma(t)| \leq |\langle v, \gamma(t) \rangle| \leq |v||\gamma(t)|$ for $t \in [0, 1)$ which shows that $\nu(\langle v, \gamma \rangle) = \nu(\gamma)$. Hence $\nu(\langle v, \gamma' \rangle) = \nu(\langle v, \gamma \rangle') = \nu(\langle v, \gamma \rangle) - 1 = \nu(\gamma) - 1 = \nu(\gamma')$.

Since $\gamma((0, 1))$ is a subset of the polar curve Γ_v, $\operatorname{grad} f(\gamma(t))$ is parallel to v for $t \in (0, 1)$. Therefore we have $|\langle \operatorname{grad} f(\gamma(t)), \gamma'(t) \rangle| = (|\operatorname{grad} f(\gamma(t))|/|v|)|\langle v, \gamma'(t) \rangle|$ for $t \in [0, 1)$. By Property 2.1 we get $\nu(\langle \operatorname{grad} f \circ \gamma, \gamma' \rangle) = \nu(\operatorname{grad} f \circ \gamma) + \nu(\langle v, \gamma' \rangle) = \nu(\operatorname{grad} f \circ \gamma) + \nu(\gamma')$ which completes the proof of the claim and the proof of the theorem. □

To prove Theorems 1.4 and 1.5 we need to estimate the growth of a gradient on polar curves. It is done in Theorem 2.5. We keep notation of Theorem 1.3.

Theorem 2.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function of degree d with an isolated zero at the origin. Then there exists an analytic curve $\gamma : (-1, 1) \to \mathbb{R}^n$, $\gamma(0) = 0$ such that: $\nu(\gamma) \leq (d - 1)^{n-1}$, $\nu(\operatorname{grad} f \circ \gamma) \leq (d - 1)^n$ and $\beta_0 = \nu(\operatorname{grad} f \circ \gamma)/\nu(\gamma)$.*

I hope that the above estimates can be improved. In this way we would obtain sharper versions of Theorems 1.4 and 1.5.

The proof is based on Lemmas 2.6 and 2.8. Let us denote $\partial_i f = \partial f/\partial x_i$.

Lemma 2.6. *Let $f : U \to \mathbb{R}$ be an analytic function defined in a neighborhood of zero $U \subset \mathbb{R}^n$ such that $\operatorname{grad} f(x) \neq 0$ for $x \in U \setminus \{0\}$ and let L be a proper linear subspace of $\mathbb{R}^n$. Then there exists $v = (v_1, \ldots, v_{n-1}, 1) \in \mathbb{R}^n \setminus L$ such that:*

(i) $\Gamma_v = \{\, x \in U : (\partial_1 f - v_1\partial_n f) = \cdots = (\partial_{n-1} f - v_{n-1}\partial_n f) = 0 \,\}$,

(ii) *the derivatives* $\mathrm{d}_x(\partial_1 f - v_1\partial_n f)$, $\ldots$, $\mathrm{d}_x(\partial_{n-1} f - v_{n-1}\partial_n f)$ *are linearly independent for all* $x \in \Gamma_v \setminus \{0\}$.

Proof. Consider the map

$$G : U \setminus (\partial_n f)^{-1}(0) \ni x \to (\partial_1 f(x)/\partial_n f(x), \ldots, \partial_{n-1} f(x)/\partial_n f(x)) \in \mathbb{R}^{n-1}.$$

The set $\{\, (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1} : (v_1, \ldots, v_{n-1}, 1) \in L \,\}$ has measure zero in $\mathbb{R}^{n-1}$. By Sard's theorem there exists a regular value $v' = (v_1, \ldots, v_{n-1})$ of G which belongs to the complement of this set. Set $v = (v_1, \ldots, v_{n-1}, 1)$. It is easy to check that Γ_v is given by (i) and $G^{-1}(v') = \Gamma_v \setminus \{0\}$. Since v' is a regular value of G, the derivatives $\mathrm{d}_x(\partial_1 f/\partial_n f)$, $\ldots$, $\mathrm{d}_x(\partial_{n-1} f/\partial_n f)$ are linearly independent for all $x \in \Gamma_v \setminus \{0\}$. By the rule of differentiating a quotient and by (i) we get $\mathrm{d}_x(\partial_i f/\partial_n f) = (1/\partial_n f)\mathrm{d}_x(\partial_i f - v_i\partial_n f)$ for $i = 1, \ldots, n-1$. Therefore $\mathrm{d}_x(\partial_1 f - v_1\partial_n f)$, $\ldots$, $\mathrm{d}_x(\partial_{n-1} f - v_{n-1}\partial_n f)$ are also linearly independent. $\square$

Notice that, in fact, we proved that for almost all $v \in \mathbb{R}^n$ (in the sense of measure theory) either Γ_v is a one dimensional analytic set or $\Gamma_v = \{0\}$. We show below that the second possibility cannot occur. This explains why we call the sets Γ_v polar curves.

Lemma 2.7. *Under the assumptions of Theorem 1.3,* 0 *is an accumulation point of* Γ_v *for every* $v \in \mathbb{R}^n \setminus \{0\}$.

Proof. We can show, using the curve–selection lemma, that for all x sufficiently close to the origin the vectors $\operatorname{grad} f(x)$ and x do not point in opposite directions (see e.g. proof of Proposition 3.8.8 in [BeR]). Let $S_r = \{x \in \mathbb{R}^n : |x| = r\}$ be a sphere of sufficiently small radius. By the previous remark the mapping $H : S_r \times [0,1] \to S_1$ given by

$$H(x,t) = \frac{(1-t)\operatorname{grad} f(x) + tx}{|(1-t)\operatorname{grad} f(x) + tx|}$$

is well defined. H is a homotopy between $H_0(x) = \operatorname{grad} f(x)/|\operatorname{grad} f(x)|$ and $H_1(x) = x/|x|$. Hence the mapping H_0 has a topological degree 1 and thus is surjective. Since we have an inclusion $H_0^{-1}(v/|v|) \subset \Gamma_v \cap S_r$, the origin is an accumulation point of Γ_v. $\square$

Lemma 2.8. *Let* $A \subset \mathbb{R}^n$ *be a real algebraic set given by equations* $H_1(x) = \cdots = H_{n-1}(x) = 0$, *where* H_1, $\ldots$, H_n *are polynomials, and let* $\psi : (-1,1) \to A$, $\psi(0) = 0$, $\psi \neq 0$ *be an analytic curve. Assume that*

(i) *the derivatives* $\mathrm{d}_x H_1$, $\ldots$, $\mathrm{d}_x H_{n-1}$ *are linearly independent for all* $x \in A \setminus \{0\}$ *in a neighborhood of zero,*

(ii) $0 \in \mathbb{R}^n$ *is an isolated point of the set* $A \cap \{ x \in \mathbb{R}^n : H_n(x) = 0 \}$.
Then there exist an analytic curve $\gamma : (-1, 1) \to A$, $\gamma(0) = 0$ *and an analytic function* s, $s(0) = 0$ *such that*

(iii) $\psi = \gamma \circ s$ *in a neighborhood of zero,*

(iv) $\nu(\gamma) \leq \prod_{i=1}^{n-1} \deg H_i$,

(v) $\nu(H_n \circ \gamma) \leq \prod_{i=1}^{n} \deg H_i$.

Proof. Regard $\mathbb{R}^n$ as a subset of $\mathbb{C}^n$. Then $A = A^{\mathbb{C}} \cap \mathbb{R}^n$ where $A^{\mathbb{C}} = \{ z \in \mathbb{C}^n : H_1(z) = \cdots = H_{n-1}(z) = 0 \}$. Let $A^{\mathbb{C}} = A_1 \cup \cdots \cup A_s$ be the decomposition of $A^{\mathbb{C}}$ into irreducible algebraic components.

Since $\psi((-1, 1)) \subset A$, there exists a component $C = A_i$ $(1 \leq i \leq s)$ of the set $A^{\mathbb{C}}$ for which $\psi((-1, 1)) \subset C$. The component C is a complex algebraic curve. Indeed by (i) there is a point $x = \psi(t)$ $(0 < t < 1)$ for which the derivatives $\mathrm{d}_x H_1, \ldots, \mathrm{d}_x H_{n-1}$ are linearly independent. Therefore $\dim_{\mathbb{C}} C \leq n - \operatorname{rank}(C, x) \leq n - \operatorname{rank}(\mathrm{d}_x H_1, \ldots, \mathrm{d}_x H_{n-1}) = 1$ (see [Wh]). Since C contains an analytic branch, $\dim_{\mathbb{C}} C = 1$.

According to Puiseux' theorem (see [Ło, 173–176]) the curve C is in a neighborhood of zero a finite union of branches. We have $C \cap U = \gamma_1(D) \cup \cdots \cup \gamma_l(D)$, where U is a neighborhood of $0 \in \mathbb{C}^n$, $D = \{t \in \mathbb{C} : |t| < 1\}$ is a unit disc and $\gamma_i : (D, 0) \to (C, 0)$ $(1 \leq i \leq l)$ are injective holomorphic curves. Moreover, according to Milnor (see [Mi] remarks after lemma 3.3) we can additionally assume that for $i = 1, \ldots, l$ if $\gamma_i(t) \in \mathbb{R}^n$ then $t \in \mathbb{R}$. The curve ψ extends to a local holomorphic (not necessarily injective) parametrization of one of branches described above, say $\gamma_1(D)$. Now it is easily seen that we can put $\gamma(t) = \gamma_1(t)$ for $t \in (-1, 1)$ and find an analytic substitution s such that $\psi(t) = \gamma(s(t))$ for small t.

Claim 1. If $F \in \mathbb{C}[X_1, \ldots, X_n]$ is a polynomial for which $F \circ \gamma \neq 0$, then $\nu(F \circ \gamma) \leq (\deg C)(\deg F)$.

Proof. In order to prove the claim we use some intersection theory. Assume that F is irreducible. Then by [Sh, 190–194] the intersection multiplicity at zero of the curve C and the hypersurface $\{F = 0\}$ is given by the formula

$$\iota_0(C, \{F = 0\}) = \sum_{i=1}^{l} \nu(F \circ \gamma_i)$$

where γ_i are injective holomorphic parametrizations of the branches of C at zero. Hence $\nu(F \circ \gamma) \leq \iota_0(C, \{F = 0\})$. By Bezout's theorem $\iota_0(C, \{F = 0\}) \leq (\deg C)(\deg F)$. Therefore $\nu(F \circ \gamma) \leq (\deg C)(\deg F)$.

If F is a reducible polynomial then the formula $\nu(F \circ \gamma) \leq (\deg C)(\deg F)$ follows from inequalities $\nu(F_i \circ \gamma) \leq (\deg C)(\deg F_i)$, where F_i are irreducible factors of F.

Claim 2. $\deg C \leq \prod_{i=1}^{n-1} \deg H_i$.

Proof. Let us recall an invariant δ of algebraic sets introduced in Łojasiewicz's book [Ło, 419–420]: *Let $W = W_1 \cup \cdots \cup W_s$ be a decomposition of an algebraic set W into irreducible components. Then, by definition $\delta(W) = \sum_{i=1}^{s} \deg W_i$.* We will use the following inequality $\delta(W \cap V) \leq \delta(W)\delta(V)$ (see [Ło]). Applying this property to the set $A^{\mathbb{C}}$ we see that $\deg C \leq \delta(A^{\mathbb{C}}) = \delta(\{H_1 = 0\} \cap \cdots \cap \{H_{n-1} = 0\}) \leq \prod_{i=1}^{n-1} \delta(\{H_i = 0\}) \leq \prod_{i=1}^{n-1} \deg H_i$. The claim follows.

Proof of (iv). Let L be a linear form such that $L \circ \gamma \neq 0$. By Claims 1 and 2 $\nu(\gamma) \leq \nu(L \circ \gamma) \leq (\deg C)(\deg L) \leq \prod_{i=1}^{n-1} \deg H_i$.

Proof of (v). By Claims 1 and 2 $\nu(H_n \circ \gamma) \leq (\deg C)(\deg H_n) \leq \prod_{i=1}^{n} \deg H_i$.

Proof of Theorem 2.5. Let $L \subset \mathbb{R}^n$ be a proper linear subspace from Theorem 1.3. By Lemma 2.6 we can take $v = (v_1, \ldots, v_{n-1}, 1) \in \mathbb{R}^n \setminus L$ such that the polar curve Γ_v satisfies conditions (i) and (ii) of 2.6 in a neighborhood of zero. Moreover there exists an analytic curve $\psi : (-1, 1) \to \Gamma_v$, $\psi(0) = 0$ such that $\beta_0 = \nu(\operatorname{grad} f \circ \psi)/\nu(\psi)$. Put $H_1 = \partial_1 f - v_1 \partial_n f, \ldots, H_{n-1} = \partial_{n-1} f - v_{n-1} \partial_n f$, $H_n = \partial_n f$. By Lemma 2.8 applied to Γ_v and ψ we see that there exists an analytic curve $\gamma : (-1, 1) \to \Gamma_v$ and an analytic substitution s such that $\psi = \gamma \circ s$ in a neighborhood of zero. Moreover $\nu(\gamma) \leq \prod_{i=1}^{n-1} \deg H_i \leq (d-1)^{n-1}$ and $\nu(H_n \circ \gamma) \leq \prod_{i=1}^{n} \deg H_i \leq (d-1)^n$. Since $\nu(\psi) = \nu(\gamma)\nu(s)$ and $\nu(\operatorname{grad} f \circ \psi) = \nu(\operatorname{grad} f \circ \gamma)\nu(s)$, the Łojasiewicz exponent β_0 equals $\nu(\operatorname{grad} f \circ \gamma)/\nu(\gamma)$.

A map $H = (H_1, \ldots, H_n)$ is a composition of $\operatorname{grad} f$ with a linear automorphism. Hence $\nu(\operatorname{grad} f \circ \gamma) = \nu(H \circ \gamma)$. Since $H \circ \gamma = (0, \ldots, 0, H_n \circ \gamma)$, $\nu(\operatorname{grad} f \circ \gamma) = \nu(H_n \circ \gamma) \leq (d-1)^n$. The theorem follows. $\square$

Proof of Theorem 1.4. Let $\sum f_\mu x^\mu$ be the Taylor series at zero of f (μ is the multi–index). Set $F(x) = \sum_{|\mu| \leq \alpha_0} f_\mu x^\mu$.

Claim 1. The polynomial F has an isolated zero at the origin. The Łojasiewicz exponent $\bar{\alpha}_0$ for the inequality $|F(x)| \geq C|x|^{\bar{\alpha}}$ is equal to α_0.

Proof of claim. Denote $[\alpha_0]$ the integer part of α_0 and set $h = f - F$. Since the order of h is greater than or equal to $[\alpha_0] + 1$, we have $|h(x)| \leq M|x|^{[\alpha_0]+1}$ for some $M > 0$ and all sufficiently small $|x|$. By Corollary 2.4 there exists $C > 0$ such that $|f(x)| \geq C|x|^{\alpha_0}$ in a neighborhood of zero.

From the above inequalities we get $|F(x)| = |f(x) - h(x)| \geq |f(x)| - |h(x)| \geq C|x|^{\alpha_0} - M|x|^{[\alpha_0]+1} = (C - M|x|^{[\alpha_0]+1-\alpha_0})|x|^{\alpha_0}$ for small $|x|$. Since $M|x|^{[\alpha_0]+1-\alpha_0} \leq 1/2C$ for sufficiently small $|x|$, we have an estimate $|F(x)| \geq 1/2C|x|^{\alpha_0}$ in a neighborhood of zero. This proves that the polynomial F has an isolated zero at the origin and shows that $\bar{\alpha}_0 \leq \alpha_0$. In order to verify that $\alpha_0 \leq \bar{\alpha}_0$ it is sufficient to change the role of f and F in the above consideration.

By Claim 1, Theorem 2.5 and Theorem 1.3 there exists an analytic curve γ

for which $\nu(\gamma) \le N^{n-1}$ for $N = [\alpha_0] - 1$ such that $\beta_0 = \alpha_0 - 1 = \bar{\alpha}_0 - 1 = \nu(\operatorname{grad} F \circ \gamma)/\nu(\gamma)$. Therefore β_0 is a rational number in the interval $[N, N+1)$ with the denominator $\le N^{n-1}$. □

Proof of Theorem 1.5. It follows from Theorem 2.5 that there exists an analytic curve γ such that $\beta_0 = \nu(\operatorname{grad} F \circ \gamma)/\nu(\gamma)$ and $\nu(\operatorname{grad} F \circ \gamma) \le (\deg F - 1)^n$. Hence $\beta_0 \le (\deg F - 1)^n$. □

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