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## Intersection homology of toric varieties and a conjecture of Kalai

Tom Braden and Robert MacPherson

**Abstract.** We prove an inequality, conjectured by Kalai, relating the  $g$ -polynomials of a polytope  $P$ , a face  $F$ , and the quotient polytope  $P/F$ , in the case where  $P$  is rational. We introduce a new family of polynomials  $g(P, F)$ , which measures the complexity of the part of  $P$  “far away” from the face  $F$ ; Kalai’s conjecture follows from the nonnegativity of these polynomials. This nonnegativity comes from showing that the restriction of the intersection cohomology sheaf on a toric variety to the closure of an orbit is a direct sum of intersection homology sheaves.

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Suppose that a  $d$ -dimensional convex polytope  $P \subset \mathbb{R}^d$  is *rational*, i.e. its vertices have all coordinates rational. Then  $P$  gives rise to a polynomial  $g(P) = 1 + g_1(P)q + g_2(P)q^2 + \dots$  with non-negative coefficients as follows. Let  $X_P$  be the *associated toric variety* (see §6 – our variety  $X_P$  is  $d + 1$ -dimensional and affine). The coefficient  $g_i(P)$  is the rank of the  $2i$ -th intersection cohomology group of  $X_P$ .

The polynomial  $g(P)$  turns out to depend only on the face lattice of  $P$ , (see §1). It can be thought of as a measure of the complexity of  $P$ ; for example,  $g(P) = 1$  if and only if  $P$  is a simplex.

Suppose that  $F \subset P$  is a face of dimension  $k < d$ . We construct an associated polytope  $P/F$  as follows: choose an  $(d - k - 1)$ -plane  $L$  whose intersection with  $P$  is a single point  $p$  of the interior of  $F$ . Let  $L'$  be a small parallel displacement of  $L$  that intersects the interior of  $P$ . The quotient  $P/F$  is the intersection of  $P$  with  $L'$ ; it is only well-defined up to a projective transformation, but its combinatorial type is well-defined (Formally we put  $P/P$  to be the empty polytope). Faces of  $P/F$  are in one-to-one correspondence with faces of  $P$  which contain  $F$ .

In Corollary 6, we show that

$$g(P) \geq g(F)g(P/F)$$

holds, coefficient by coefficient. This was conjectured by Kalai in [11], where some of its applications were discussed. The special case of the linear and quadratic

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terms was proved in [12]. Roughly, this inequality means that the complexity of  $P$  is bounded from below by the complexity of the face  $F$  and the normal complexity  $g(P/F)$  to the face  $F$ .

The principal idea is to introduce *relative  $g$ -polynomials*  $g(P, F)$  for any face  $F$  of  $P$  (§2). These generalize the ordinary  $g$ -polynomials since  $g(P, P) = g(P)$ . They are also combinatorially determined by the face lattice. They measure the complexity of  $P$  relative to the complexity of  $F$ . For example, if  $P$  is the join of  $F$  with another polytope, then  $g(P, F) = 1$  (the converse, however, does not hold).

Our main result gives an interpretation of the coefficients  $g_i(P, F)$  of the relative  $g$ -polynomials as dimensions of vector spaces arising from the topology of the toric variety  $X_P$ . This shows that the coefficients are positive. Kalai's conjecture is a corollary.

The combinatorial definition of the relative  $g$ -polynomials  $g(P, F)$  makes sense whether or not the polytope  $P$  is rational. We conjecture that  $g(P, F) \geq 0$  for any polytope  $P$ ; this would imply Kalai's conjecture for general polytopes.

This paper is organized as follows: The first three sections are entirely about the combinatorics of polyhedra. They develop the properties of relative  $g$ -polynomials as combinatorial objects, with the application to Kalai's conjecture. The last three sections concern algebraic geometry. A separate guide to their contents is included in the introduction to §§4 - 6.

## 1. $g$ -numbers of polytopes

Let  $P \subset \mathbb{R}^d$  be a  $d$ -dimensional convex polytope, i.e. the convex hull of a finite collection of points affinely spanning  $\mathbb{R}^d$ . The set of faces of  $P$ , ordered by inclusion, forms a poset which we will denote by  $\mathcal{F}(P)$ . We include the empty face  $\emptyset = \emptyset_P$  and  $P$  itself as members of  $\mathcal{F}(P)$ . It is a graded poset, with the grading given by the dimension of faces. By convention we set  $\dim \emptyset = -1$ . Faces of  $P$  of dimension 0, 1, and  $d - 1$  will be referred to as vertices, edges, and facets, respectively.

Given a face  $F$  of  $P$ , the poset  $\mathcal{F}(F)$  is isomorphic to the interval  $[\emptyset, F] \subset \mathcal{F}(P)$ . The interval  $[F, P]$  is the face poset of the polytope  $P/F$  defined in the introduction.

Given the polytope  $P$ , there are associated polynomials (first introduced in [14])  $g(P) = \sum g_i(P)q^i$  and  $h(P) = \sum h_i(P)q^i$ , defined recursively as follows:

- $g(\emptyset) = 1$
- $h(P) = \sum_{\emptyset \leq F \leq P} (q - 1)^{\dim P - \dim F - 1} g(F)$ , and
- $g_0(P) = h_0(P)$ ,  $g_i(P) = h_i(P) - h_{i-1}(P)$  for  $0 < i \leq \dim P/2$ , and  $g_i(P) = 0$  for all other  $i$ .

The coefficients of these polynomials will be referred to as the  $g$ -numbers and  $h$ -numbers of  $P$ , respectively. We do not discuss the  $h$ -polynomial further in this paper.

These numbers depend only on the poset  $\mathcal{F}(P)$ . In fact, as Bayer and Billera

[1] showed, they depend only on the flag numbers of  $P$ : given a sequence of integers  $I = (i_1, \dots, i_n)$  with  $0 \leq i_1 < i_2 < \dots < i_n \leq d$ , an  $I$ -flag is an  $n$ -tuple  $F_1 < F_2 < \dots < F_n$  of faces of  $P$  with  $\dim F_k = i_k$  for all  $k$ . The  $I$ -th flag number  $f_I(P)$  is the number of  $I$ -flags. Letting  $P$  vary over all polytopes of a given dimension  $d$ , the numbers  $g_i(P)$  and  $h_i(P)$  can be expressed as a  $\mathbb{Z}$ -linear combination of the  $f_I(P)$ .

Conjecturally all the  $g_i(P)$  should be nonnegative for all  $P$ . This is known to be true for  $i = 1, 2$  [10]. For higher values of  $i$ , it can be proved for rational polytopes using the interpretation of  $g_i(P)$  as an intersection cohomology Betti number of an associated toric variety.

**Proposition 1.** *If  $P$  is a rational polytope, then  $g_i(P) \geq 0$  for all  $i$ .*

## 2. Relative $g$ -polynomials

The following proposition defines a relative version of the classical  $g$ -polynomials.

**Proposition 2.** *There is a unique family of polynomials  $g(P, F)$  associated to a polytope  $P$  and a face  $F$  of  $P$ , satisfying the following relation: for all  $P, F$ , we have*

$$\sum_{F \leq E \leq P} g(E, F)g(P/E) = g(P). \quad (1)$$

*Proof.* The equation (1) can be used inductively to compute  $g(P, F)$ , since the left hand side gives  $g(P, F) \cdot 1$  plus terms involving  $g(E, F)$  where  $\dim E < \dim P$ . The induction starts when  $P = F$ , which gives  $g(F, F) = g(F)$ .  $\square$

As an example, if  $F$  is a facet of  $P$ , then  $g(P, F) = g(P) - g(F)$ . Just as before we will denote the coefficient of  $q^i$  in  $g(P, F)$  by  $g_i(P, F)$ .

We have the following notion of relative flag numbers. Let  $P$  be a  $d$ -polytope, and  $F$  a face of dimension  $e$ . Given a sequence of integers  $I = (i_1, \dots, i_n)$  with  $0 \leq i_1 < i_2 < \dots < i_n \leq d$  and a number  $1 \leq k \leq n$  with  $i_k \geq e$ , define the relative flag number  $f_{I,k}(P, F)$  to be the number of  $I$ -flags  $(F_1, \dots, F_n)$  with  $F \leq F_k$ . Note that letting  $k = n$  and  $i_n = d$  gives the ordinary flag numbers of  $P$  as a special case. Also note that the numbers  $f_{I,k}$  where  $i_k = e$  give products of the form  $f_J(F)f_{J'}(P/F)$ , and all such products can be expressed this way.

**Proposition 3.** *Fixing  $\dim P$  and  $\dim F$ , the relative  $g$ -number  $g_i(P, F)$  is a  $\mathbb{Z}$ -linear combination of the  $f_{I,k}(P, F)$ .*

*Proof.* Use induction on  $\dim P/F$ . If  $P = F$ , then we have  $g(P, P) = g(P)$  and the result is just the corresponding result for the ordinary flag numbers. If  $P \neq F$ ,

the equation (1) gives

$$g(P, F) = g(P) - \sum_{e=\dim F}^{\dim P-1} \sum_{\substack{\dim E=e \\ F \leq E < P}} g(E, F)g(P/E).$$

For every  $e$  the coefficients of the inner summation on the right hand side are  $\mathbb{Z}$ -linear combinations of the  $f_{I,k}(P, F)$ , using the inductive hypothesis.  $\square$

The following theorem is the main result of this paper. It will be a consequence of Theorem 11.

**Theorem 4.** *If  $P$  is a rational polytope and  $F$  is any face, then  $g_i(P, F) \geq 0$  for all  $i$ .*

**Corollary 5.** (Kalai's conjecture) *If  $P$  is a rational polytope and  $F$  is any face, then*

$$g(P) \geq g(F)g(P/F),$$

where the inequality is taken coefficient by coefficient.

*Proof.* For any face  $E$  of  $P$  the polytope  $P/E$  is rational, so we have  $g(P) = g(F, F)g(P/F) +$  other nonnegative terms.  $\square$

### 3. Some examples and formulas

This section contains further combinatorial results on the relative  $g$ -polynomials. They are not used in the remainder of the paper.

First, we give an interpretation of  $g_1(P, F)$  and  $g_2(P, F)$  analogous to the ones Kalai gave for the usual  $g_1$  and  $g_2$  in [10]. We begin by recalling those results from [10].

Given a finite set of points  $V \subset \mathbb{R}^d$  define the space  $\mathcal{Aff}(V)$  of affine dependencies of  $V$  to be

$$\{ a \in \mathbb{R}^V \mid \sum_{v \in V} a_v = 0, \sum_{v \in V} a_v \cdot v = 0 \}.$$

If  $V_P$  is the set of vertices of a polytope  $P \subset \mathbb{R}^d$ , then  $\mathcal{Aff}(V_P)$  is a vector space of dimension  $g_1(P)$ .

To describe  $g_2(P)$  we need the notion of stress on a framework. A framework  $\Phi = (V, E)$  is a finite collection  $V$  of points in  $\mathbb{R}^d$  together with a finite collection  $E$  of straight line segments (edges) joining them. Given a finite set  $S$ , we denote the standard basis elements of  $\mathbb{R}^S$  by  $1_s, s \in S$ . The space of stresses  $\mathcal{S}(\Phi)$  is the kernel of the linear map

$$\alpha: \mathbb{R}^E \rightarrow \mathbb{R}^V \otimes \mathbb{R}^d,$$

defined by

$$\alpha(1_e) = 1_{v_1} \otimes (v_1 - v_2) + 1_{v_2} \otimes (v_2 - v_1),$$

where  $v_1$  and  $v_2$  are the endpoints of the edge  $e$ . A stress can be described physically as an assignment of a contracting or expanding force to each edge, such that the total force resulting at each vertex is zero.

To a polytope  $P$  we can associate a framework  $\Phi_P$  by taking as vertices the vertices of  $P$ , and as edges the edges of  $P$  together with enough extra edges to triangulate all the 2-faces of  $P$ . Then  $g_2(P)$  is the dimension of  $\mathcal{S}(\Phi_P)$ .

Given a polytope  $P$  and a face  $F$ , define the closed union of faces  $N(P, F)$  to be the union of all facets of  $P$  containing  $F$ . Note that  $N(P, \emptyset) = \partial P$ , and  $N(P, P) = \emptyset$ . Let  $V_N$  be the set of vertices of  $P$  in  $N(P, F)$ , and define a framework  $\Phi_N$  by taking all edges and vertices of  $\Phi_P$  contained in  $N(P, F)$ .

**Theorem 6.** *We have*

$$g_1(P, F) = \dim_{\mathbb{R}} \mathcal{A}ff(V_P) / \mathcal{A}ff(V_N), \text{ and}$$

$$g_2(P, F) = \dim_{\mathbb{R}} \mathcal{S}(\Phi_P) / \mathcal{S}(\Phi_N),$$

using the obvious inclusions of  $\mathcal{A}ff(V_N)$  in  $\mathcal{A}ff(V_P)$  and  $\mathcal{S}(\Phi_N)$  in  $\mathcal{S}(\Phi_P)$ .

The proof for  $g_1$  is an easy exercise; the proof for  $g_2$  will appear in a forthcoming paper [3].

Next, we have a formula which shows that  $g(P, F)$  can be decomposed in the same way  $g(P)$  was in Proposition 2. Given two faces  $E, F$  of a polytope  $P$ , let  $E \vee F$  be the unique smallest face containing both  $E$  and  $F$ .

**Proposition 7.** *For any polytope  $P$  and faces  $F' \leq F$  of  $P$ , we have*

$$g(P, F) = \sum_{F' \leq E} g(E, F')g(P/E, (E \vee F)/E).$$

*Proof.* Again, we show that this formula for  $g(P, F)$  satisfies the defining relation of Proposition 2. Fix  $F' \leq F$ , and define  $\hat{g}(P, F)$  to be the above sum. Then we have

$$\begin{aligned} \sum_{F \leq D} \hat{g}(D, F)g(P/D) &= \sum_{\substack{F' \leq E \\ F \vee E \leq D}} g(P/D)g(E, F')g(D/E, (E \vee F)/E) \\ &= \sum_{F' \leq E} g(E, F')g(P/E) \\ &= g(P). \end{aligned}$$

Since the computation of  $g(P, F)$  from Proposition 2 only involves computation of  $g(E, F)$  for other faces  $E$  of  $P$ , this proves that  $\hat{g}(P, F) = g(P, F)$ , as required.  $\square$

Finally, we can carry out the inversion implicit in Proposition 2 explicitly. First we need the notion of polar polytopes. Given a polytope  $P \subset \mathbb{R}^d$ , we can assume that the origin lies in the interior of  $P$  by moving  $P$  by an affine motion. The polar polytope  $P^*$  is defined by

$$P^* = \{x \in (\mathbb{R}^*)^d \mid \langle x, y \rangle \leq 1 \text{ for all } y \in P\}.$$

The face poset  $\mathcal{F}(P^*)$  is canonically the opposite poset to  $\mathcal{F}(P)$ . Define  $\bar{g}(P) = g(P^*)$ .

**Proposition 8.** *We have*

$$g(P, F) = \sum_{F \leq F' \leq P} (-1)^{\dim P - \dim F'} g(F') \bar{g}(P/F'). \tag{2}$$

*Proof.* We use the following formula, due to Stanley [15]: For any polytope  $P \neq \emptyset$ , we have

$$\sum_{\emptyset \leq F \leq P} (-1)^{\dim F} \bar{g}(F) g(P/F) = 0. \tag{3}$$

Now define  $\hat{g}(P, F)$  to be the right hand side of (2). We will show that the defining property (1) of Proposition 2 holds.

Pick a face  $F$  of  $P$ . We have, using (3),

$$\begin{aligned} \sum_{F \leq E \leq P} \hat{g}(E, F) g(P/E) &= \sum_{F \leq F' \leq E \leq P} (-1)^{\dim E - \dim F'} g(F') \bar{g}(E/F') g(P/E) \\ &= \sum_{F \leq F' \leq P} g(F') \sum_{F' \leq E \leq P} (-1)^{\dim E - \dim F'} \bar{g}(E/F') g(P/E) \\ &= g(P), \end{aligned}$$

as required.  $\square$

## Introduction to §§4 - 6

The remainder of the paper uses the topology of toric varieties to describe the polynomial  $g(P, F)$  when  $P$  is rational. Given  $P$ , there is an associated affine toric variety  $X_P$ , and  $g(P)$  gives the local intersection cohomology betti numbers of  $X_P$  at the unique torus fixed point  $p$ .

The main topological result is the following (Theorem 10). Let  $Y \subset X_P$  be the closure of one of the torus orbits. Then the restriction of the intersection cohomology sheaf  $\mathbf{IC}(X)$  to  $Y$  is a direct sum of intersection cohomology sheaves, with shifts, supported on subvarieties of  $Y$  (a related result is given by Victor Ginzburg in [8], Lemma 3.5). The polynomial  $g_i(P, F)$  measures the number of copies of the intersection cohomology sheaf  $\mathbf{IC}(\{p\})$  that appear with shift  $2i$  in the restriction of the intersection cohomology sheaf of  $X_P$  to  $Y_F$ , where  $Y_F$  is the closure of the orbit corresponding to the face  $F$ .

To prove Theorem 10 we construct a certain resolution (the Seifert resolution, §5)  $p: \tilde{X} \rightarrow X$  of  $X$ . Its key property is that the inclusion of  $\tilde{Y} = p^{-1}(Y)$  in  $\tilde{X}$  is “ $\mathbb{Q}$ -homology normally nonsingular” - the restriction of the intersection cohomology sheaf of  $\tilde{X}$  to  $\tilde{Y}$  is an intersection cohomology sheaf (Proposition 14).

This construction, and hence Theorem 10, work in situations other than toric varieties; essentially any variety  $X$  with a  $\mathbb{C}^*$  action contracting  $X$  onto the fixed point set  $Y$  will satisfy Theorem 10. The proof we give, while easier than the general result, only works for toric varieties.

## 4. Toric varieties

We will only sketch the properties of toric varieties that we will need. For a more complete presentation, see [7]. Throughout this section let  $P$  be a  $d$ -dimensional rational polytope in  $\mathbb{R}^d$ .

Define a toric variety  $X_P$  as follows. Embed  $\mathbb{R}^d$  into  $\mathbb{R}^{d+1}$  by

$$(x_1, \dots, x_d) \mapsto (x_1, \dots, x_d, 1),$$

and let  $\sigma = \sigma_P$  be the cone over the image of  $P$  with apex at the origin in  $\mathbb{R}^{d+1}$ . It is a rational polyhedral cone with respect to the standard lattice  $N = \mathbb{Z}^{d+1}$ . More generally, if  $F$  is a face of  $P$ , let  $\sigma_F$  be the cone over the image of  $F$ ; set  $\sigma_\emptyset = \{0\}$ .

Define  $X = X_P$  to be the affine toric variety  $X_\sigma$  corresponding to  $\sigma$ . It is the variety  $\text{Spec } \mathbb{C}[M \cap \sigma^\vee]$ , where

$$\sigma^\vee = \{\mathbf{x} \in (\mathbb{R}^{d+1})^* \mid \langle \mathbf{x}, \mathbf{y} \rangle \geq 0 \text{ for all } \mathbf{y} \in \sigma\}$$

is the dual cone to  $\sigma$ ,  $M$  is the dual lattice to  $N$ , and  $\mathbb{C}[M \cap \sigma^\vee]$  is the semigroup algebra of  $M \cap \sigma^\vee$ . It is a  $(d+1)$ -dimensional normal affine algebraic variety, on

which the torus  $T = \text{Hom}(M, \mathbb{C}^*)$  acts. Let  $f_{\mathbf{v}}: X_P \rightarrow \mathbb{C}$  be the regular function corresponding to the point  $\mathbf{v} \in M \cap \sigma^\vee$ .

The orbits of the action of  $T$  on  $X$  are parametrized by the faces of  $P$ . Let  $F$  be any face of  $P$ , including the empty face, and let

$$\sigma_F^\perp = \{ \mathbf{x} \in \sigma^\vee \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for all } \mathbf{y} \in \sigma_F \}$$

be the face of  $\sigma^\vee$  dual to  $\sigma_F$ . Then the variety

$$O_F := \{ x \in X \mid f_{\mathbf{v}}(x) \neq 0 \iff \mathbf{v} \in M \cap \sigma_F^\perp \}$$

is a  $T$ -orbit, isomorphic to the torus  $(\mathbb{C}^*)^{d-e}$ , where  $e = \dim F$ . Furthermore, all  $T$ -orbits arise this way. In particular,  $X_P$  has a unique  $T$ -fixed point  $\{p\} = O_P$ .

Given a face  $E$ , the union

$$U_E = \bigcup_{F \leq E} O_F$$

is a  $T$ -invariant open neighborhood of  $O_E$ . There is a non-canonical isomorphism  $U_E \cong O_E \times X_E$  where  $X_E$  is the affine toric variety defined by the cone  $\sigma_E$ , considered as a subset of the affine it spans, with the lattice given by restricting  $N$ . If  $O_F^E$  denotes the orbit of  $X_E$  corresponding to a face  $F \leq E$ , then  $O_F$  sits in  $U_E \cong O_E \times X_E$  as  $O_E \times O_F^E$ .

The closure of the orbit  $O_E$  is given by

$$\overline{O_E} = \bigcup_{F \geq E} O_F;$$

it is isomorphic to the affine toric variety  $X_{P/E}$ . More precisely, it is the affine toric variety corresponding to the cone  $\tau = \sigma/\sigma_E$ , the image of  $\sigma$  projected into  $\mathbb{R}^{d+1}/\text{span } \sigma_E$ , with the lattice given by the projection of  $N$ ;  $\tau$  is a cone over a polytope projectively equivalent to  $P/E$ .

The connection between toric varieties and  $g$ -numbers of polytopes is given by the following result. Proofs appear in [5, 6]. We consider the intersection cohomology sheaf  $\mathbf{IC}(X)$  of a variety  $X$  as an object in the bounded derived category  $D^b(X)$  of sheaves of  $\mathbb{Q}$ -vector spaces on  $X$ . We will take the convention that  $\mathbf{IC}(X)$  restricts to a constant local system placed in degree zero on an smooth open subset of  $X$ .

**Proposition 9.** *The local intersection cohomology groups of  $X_P$  are described as follows. Take  $x \in O_F$ , and let  $j_x$  be the inclusion. Then*

$$\dim \mathbb{H}^{2i} j_x^* \mathbf{IC}(X_P) = g_i(F),$$

and  $\mathbb{H}^k j_x^* \mathbf{IC}(X_P)$  vanishes for odd  $k$ .

**Definition.** Call an object  $\mathbf{A}$  in  $D^b(X)$  *pure* if it is a direct sum of shifted intersection cohomology sheaves

$$\bigoplus_{\alpha} \mathbf{IC}^*(Z_{\alpha}; \mathcal{L}_{\alpha})[n_{\alpha}], \tag{4}$$

where each  $Z_{\alpha}$  is an irreducible subvariety of  $X$ ,  $\mathcal{L}_{\alpha}$  is a simple local system on a Zariski open subset  $U_{\alpha}$  of the smooth locus of  $Z_{\alpha}$ , and  $n_{\alpha}$  is an integer.

Now fix a face  $F$  of  $P$ . The following theorem is the main result of this paper. It will be proved in the following two sections.

**Theorem 10.** *Let  $j: \overline{O}_F \rightarrow X_P$  be the inclusion. Then the pullback  $\mathbf{A} = j^* \mathbf{IC}^*(X_P)$  of the intersection cohomology sheaf on  $X_P$  is pure.*

As a result, since the local intersection cohomology exists only in even degrees and gives trivial local systems on the orbits  $O_Y$ , we get

$$\mathbf{A} = \bigoplus_{E \geq F} \bigoplus_{i \geq 0} \mathbf{IC}^*(\overline{O}_E)[-2i] \otimes V_E^i, \tag{5}$$

for some finite dimensional  $\mathbb{Q}$ -vector spaces  $V_E^i$ .

Now we can give an interpretation of the combinatorially defined polynomials  $g(P, F)$  for rational polytopes which implies nonnegativity, and hence Theorem 4. Let  $\{p\} = O_P$  be the unique  $T$ -fixed point of  $X_P$ .

**Theorem 11.** *The relative  $g$ -number  $g_i(P, F)$  is given by*

$$g_i(P, F) = \dim_{\mathbb{Q}} V_P^i.$$

*Proof.* Taking this for the moment as a definition of  $g(P, F)$ , we will show that the defining relation of Proposition 2 holds. It will be enough to show that  $\dim_{\mathbb{Q}} V_E^i = g(E, F)$  for a face  $F \leq E \neq P$ , since then taking the dimensions of the stalk cohomology groups on both sides of (5) gives exactly the desired relation (1).

Consider the commutative diagram of inclusions

$$\begin{array}{ccc} \overline{O}_F^E & \xrightarrow{j'} & X_E \\ k' \downarrow & & \downarrow k \\ \overline{O}_F & \xrightarrow{j} & X_P \end{array}$$

where  $k$  maps  $X_E \cong \{x\} \times X_E$  into  $O_E \times X_E \cong U_E \subset X_P$ , and  $k'$  is the restriction of  $k$ .

Then  $k$  is a normally nonsingular inclusion, so we have

$$(j')^*k^*\mathbf{IC}(X_P) = (j')^*\mathbf{IC}(X_E) = \bigoplus_{F \leq F' \leq E} \bigoplus_{i \geq 0} \mathbf{IC}(\overline{O_{F'}^E})[-2i] \otimes W_{F'}^i$$

for some vector spaces  $W_{F'}^i$ . On the other hand, since  $k'$  is a normally nonsingular inclusion, it is also equal to

$$(k')^*j^*\mathbf{IC}(X_P) = \bigoplus_{F \leq F' \leq E} \bigoplus_{i \geq 0} \mathbf{IC}(\overline{O_{F'}^E})[-2i] \otimes V_{F'}^i.$$

Where the  $V_{F'}^i$  are as in (5).

Comparing terms, we see that  $W_E^i \cong V_E^i$ , so we have

$$\dim_{\mathbb{Q}} V_E^i = \dim_{\mathbb{Q}} W_E^i = g_i(E, F),$$

as required. □

### 5. The Seifert resolution

Fix a face  $F$  of the polytope  $P$ , and let  $\tau = \sigma_F$ ,  $X = X_P$ ,  $Y = Y_F$ . Our proof of Theorem 10 involves constructing a certain resolution  $\tilde{X}$  of  $X$ , which we call a *Seifert resolution* of the pair  $(X, Y)$ . First we need to choose an action of  $\mathbb{C}^*$  on  $X$  for which  $Y$  is the fixed-point set.

Let  $\mathbf{a}$  be any lattice point in the relative interior of the cone  $\tau$ . Define the rank-one subtorus  $T_{\mathbf{a}} \subset T \cong \text{Hom}(M, \mathbb{C}^*)$  to be the kernel of the restriction

$$\text{Hom}(M, \mathbb{C}^*) \rightarrow \text{Hom}(M \cap \mathbf{a}^\perp, \mathbb{C}^*).$$

The map  $M \rightarrow \mathbb{Z}$  given by pairing with  $\mathbf{a}$  defines a homomorphism  $\mathbb{C}^* = \text{Hom}(\mathbb{Z}, \mathbb{C}^*) \rightarrow T = \text{Hom}(M, \mathbb{C}^*)$  with image contained in  $T_{\mathbf{a}}$ , thus defining an action of  $\mathbb{C}^*$  on  $X$ .

**Proposition 12.**  *$Y$  is the fixed-point set of this action, and for any  $x \in X$  we have*

$$\lim_{t \rightarrow 0} t \cdot x \in Y.$$

We say that  $Y$  is an *attractor* for the  $\mathbb{C}^*$  action.

Let  $X^\circ = X \setminus Y$ . By the proposition above, the map  $X^\circ \times \mathbb{C}^* \rightarrow X^\circ$  defined by our  $\mathbb{C}^*$  action extends to a map  $X^\circ \times \mathbb{C} \rightarrow X$ . Let  $\tilde{X}$  be the quotient  $X^\circ \times \mathbb{C} / \sim$ , where the equivalence relation is given by

$$(x, s) \sim (t \cdot x, t^{-1}s)$$

for  $t \in \mathbb{C}^*$ . There is an induced map  $p: \tilde{X} \rightarrow X$ . We can let  $T$  act on  $X^\circ \times \mathbb{C}$  by acting on the first factor; this action passes to  $\tilde{X}$ , and  $p$  is an equivariant map. Let  $\tilde{Y} = p^{-1}(Y)$ ,  $\tilde{X}^\circ = p^{-1}(X^\circ)$ .

**Proposition 13.** *The map  $p$  is proper, and restricts to an isomorphism  $\tilde{X}^\circ \cong X^\circ$ . Furthermore,  $\tilde{Y} \cong (X^\circ \times \{0\})/\mathbb{C}^*$  is a divisor in  $\tilde{X}$ , and is an attractor for the  $\mathbb{C}^*$  action on  $\tilde{X}$  defined by the lattice point  $a$ .*

We call the pair  $(\tilde{X}, \tilde{Y})$  a Seifert resolution of  $(X, Y)$ . The action of  $T$  makes  $\tilde{X}$  into a toric variety. An explicit description of its fan will be useful. Take a fan consisting of all cones of the form  $\rho$  and  $\rho_{\mathbf{a}} = \rho + \mathbb{R}_{\geq 0}\mathbf{a}$ , where  $\rho$  runs over all faces of  $\sigma$  which do not contain  $\tau$ . Then  $\tilde{X}$  is the toric variety defined by this fan, and  $\tilde{Y}$  is the union of the orbits corresponding to the cones  $\rho_{\mathbf{a}}$ .

The inclusion  $\tilde{j}: \tilde{Y} \rightarrow \tilde{X}$  looks almost like the inclusion of the zero section of a line bundle; for instance, if  $X$  is conical,  $Y = \{p\}$  is the cone point and  $a$  is chosen to give the conical  $\mathbb{C}^*$  action, then  $\tilde{X}$  is just the blow-up of  $X$  along  $Y$ .

**Proposition 14.** *There is an isomorphism*

$$\tilde{j}^* \mathbf{IC}(\tilde{X}) \cong \mathbf{IC}(\tilde{Y}).$$

We will prove this in the next section; first, we show how it implies Theorem 10. Consider the fiber square

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{j}} & \tilde{X} \\ \downarrow q & & \downarrow p \\ Y & \xrightarrow{j} & X \end{array}$$

where  $q = p|_{\tilde{Y}}$ . Because  $p$  and  $q$  are proper we have

$$Rq_* \tilde{j}^* \mathbf{IC}(\tilde{X}) \cong j^* Rp_* \mathbf{IC}(\tilde{X}).$$

The left hand side is  $Rq_* \mathbf{IC}(\tilde{Y})$  by Proposition 14, which is pure by the decomposition theorem [2]. The decomposition theorem also implies that  $\mathbf{A} = Rp_* \mathbf{IC}(\tilde{X})$  is pure, and because  $\tilde{X} \rightarrow X$  is an isomorphism on a Zariski dense subset, the intersection cohomology sheaf of  $X$  must occur in  $\mathbf{A}$  with zero shift. Thus the right hand side becomes

$$j^*(\mathbf{IC}(X)) \oplus j^* \mathbf{A}'$$

where  $\mathbf{A}'$  is pure. Theorem 10 now follows from the following lemma.

**Lemma 15.** *If  $\mathbf{A}, \mathbf{B}$  are objects in  $D^b(X)$  and  $\mathbf{A} \oplus \mathbf{B}$  is pure, then so is  $\mathbf{A}$ .*

*Proof.* Denote  $\mathbf{A} \oplus \mathbf{B}$  by  $\mathbf{C}$ . Since  $\mathbf{C}$  is pure, it is isomorphic to the direct sum

$$\bigoplus_{i \in \mathbb{Z}} {}^p H^i(\mathbf{C})[-i]$$

of its perverse homology sheaves. Each  ${}^p H^i(\mathbf{C}) = {}^p H^i(\mathbf{A}) \oplus {}^p H^i(\mathbf{B})$  is a pure perverse sheaf, and since the category of perverse sheaves is abelian,  ${}^p H^i(\mathbf{A})$  is pure. Then the composition

$$\bigoplus {}^p H^i(\mathbf{A})[-i] \rightarrow \bigoplus {}^p H^i(\mathbf{C})[-i] \cong \mathbf{C} \rightarrow \mathbf{A}$$

induces an isomorphism on all the perverse homology sheaves, and hence is an isomorphism (see [2], §1.3).  $\square$

**6. Proof of Proposition 14**

Let  $\mathbf{A} = \tilde{j}^* \mathbf{IC}(\tilde{X})$ . We will show that  $\mathbf{A}$  satisfies the vanishing conditions for intersection cohomology on the stalk and costalk cohomology groups [9], and thus must be isomorphic to  $\mathbf{IC}(\tilde{Y})$ .

If  $\tilde{X}$  is a line bundle over  $\tilde{Y}$ , the result is immediate. In general, we can take a quotient by a finite group which acts trivially on  $\tilde{Y}$  and get a line bundle. This works for more general varieties than toric varieties, but for our purposes a combinatorial proof will suffice.

We continue the notation of the previous section. For each face  $\rho$  not containing  $\tau$ , let  $n_\rho$  be the index of the lattice  $(N \cap \text{span}(\rho)) + \mathbf{a}\mathbb{Z}$  in  $N$ . If  $n = \text{lcm } n_\rho$ , then we can define a lattice  $N' = N + (\mathbf{a}/n)\mathbb{Z}$  containing  $N$ . We get a corresponding map of tori  $T \rightarrow T'$ ; the kernel  $G$  is a finite cyclic group inside  $T_{\mathbf{a}}$ .

**Proposition 16.** *The quotient  $\tilde{X}/G$  is a line bundle over  $\tilde{Y}/G \cong \tilde{Y}$ .*

Using this, we prove Proposition 14. We can retract  $\tilde{X}$  onto  $\tilde{Y}$  using the  $\mathbb{C}^*$  action; we get an isomorphism

$$\mathbf{A} \cong R\pi_* \mathbf{IC}(\tilde{X}),$$

where  $\pi: \tilde{X} \rightarrow \tilde{Y}$  is the projection defined by the action.

For a point  $y \in \tilde{Y}$ , we can find a neighborhood  $N \subset \tilde{Y}$  of  $y$  so that the stalk and costalk cohomology groups of  $\mathbf{A}$  are given by

$$\begin{aligned} \mathbb{H}^i i_y^* \mathbf{A} &= IH_{n-i}(\pi^{-1}(N), \pi^{-1}(\partial N)), \\ \mathbb{H}^i i_y^! \mathbf{A} &= IH_{n-i}(\pi^{-1}(N)). \end{aligned}$$

Since  $G \subset T_{\mathbf{a}}$ , elements of  $G$  preserve the fibers of  $\pi$  and act by transformations which are isotopic to the identity. Thus  $G$  acts trivially on the stalks and costalks of  $\mathbf{A}$ . The following lemma then shows that they are isomorphic to  $IH_{n-i}(\pi^{-1}(N)/G, \pi^{-1}(\partial N)/G)$  and  $IH_{n-i}(\pi^{-1}(N)/G)$ , respectively, and hence to  $IH_{n-i}(N, \partial N)$  and  $IH_{n-i}(N)$ , since  $\tilde{X}/G$  is a line bundle over  $\tilde{Y}$ . The required vanishing follows immediately.

**Lemma 17.** *Let  $X$  be a pseudomanifold, acted on by a finite group  $G$ , and let  $Y$  be a  $G$ -invariant subspace. Then there is an isomorphism*

$$IH_*(X/G, Y/G; \mathbb{Q}) \cong IH_*(X, Y; \mathbb{Q})^G$$

*between the intersection homology of the pair  $(X/G, Y/G)$  and the  $G$ -stable part of the intersection homology of  $(X, Y)$ .*

*Proof.* Give  $X$  a  $G$ -invariant triangulation. Then the intersection homology of  $X$  can be expressed by means of simplicial chains of the barycentric subdivision, see [13, Appendix]. Now the standard argument in [4, p. 120] can be applied.  $\square$

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