

Open manifolds with nonnegative Ricci curvature and large volume growth

Autor(en): **Xia, Changyu**

Objektyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **74 (1999)**

PDF erstellt am: **22.07.2024**

Persistenter Link: <https://doi.org/10.5169/seals-55796>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Open manifolds with nonnegative Ricci curvature and large volume growth

Changyu Xia

Abstract. In this paper, we study complete open n -dimensional Riemannian manifolds with nonnegative Ricci curvature and large volume growth. We prove among other things that such a manifold is diffeomorphic to a Euclidean n -space R^n if its sectional curvature is bounded from below and the volume growth of geodesic balls around some point is not too far from that of the balls in R^n .

Mathematics Subject Classification (1991). (1985 *Revision*): 53C20; Secondary 53C21, 53R70, 31C12.

Keywords. Open manifolds, nonnegative Ricci curvature, large volume growth.

1. Introduction

Let (M, g) be an n -dimensional complete Riemannian manifold with nonnegative Ricci curvature. The relative volume comparison theorem [BC, GLP] says that the function $r \rightarrow \frac{\text{vol}[B(p, r)]}{\omega_n r^n}$ is monotone decreasing, where $B(p, r)$ denotes the geodesic ball around $p \in M$ with radius r and ω_n is the volume of the unit ball in the Euclidean space R^n . Define α_M by

$$\alpha_M = \lim_{r \rightarrow \infty} \frac{\text{vol}[B(p, r)]}{\omega_n r^n}.$$

It is easy to show that α_M is independent of $p \in M$, hence it is a global geometric invariant of M . We always have

$$\alpha_M \omega_n r^n \leq \text{vol}[B(x, r)] \leq \omega_n r^n, \quad \forall r > 0, \quad \forall x \in M. \quad (1.1)$$

We say (M, g) has *large volume growth* if $\alpha_M > 0$. It should be noticed that, in this case, $0 < \alpha_M \leq 1$ and when $\alpha_M = 1$, M is isometric to R^n by Bishop-Gromov comparison theorem [BC, GLP].

A manifold M is said to have finite topological type if there is a compact domain Ω whose boundary $\partial\Omega$ is a topological manifold such that $M \setminus \Omega$ is homeomorphic

to $\partial\Omega \times [0, \infty)$. Abresch-Gromoll [AG] first obtain the finiteness of topological type for complete n -manifolds (M, g) with $\text{Ric}_M \geq 0$ and small diameter growth $\text{diam}(p, r) = o(\frac{1}{r^n})$, provided that the sectional curvature $K_M \geq K_0 > -\infty$.

Let (M, g) be an n -dimensional complete manifold with $\text{Ric}_M \geq 0$ and $\alpha_M > 0$. It has been proved by Li [L] that M has finite fundamental group. Anderson [A] has showed that the order of the fundamental group of M is bounded from above by $\frac{1}{\alpha_M}$. Perelman [P] has proved that there is a small constant $\epsilon(n) > 0$ depending only on n such that if $\alpha_M > 1 - \epsilon(n)$, then M is contractible. It has been shown by Shen [S2] that M has finite topological type, provided that $\frac{\text{vol}[B(p,r)]}{\omega_n r^n} = \alpha_M + o(\frac{1}{r^{n-1}})$ and, either the conjugate radius $\text{conj}_M \geq c > 0$ or the sectional curvature $K_M \geq K_0 > -\infty$. Petersen [Pe] conjectured that if $\alpha_M > \frac{1}{2}$ then M is diffeomorphic to R^n . Recently, Cheeger and Colding [CC] gave a partial answer to Petersen's conjecture. In fact, they proved that there exists a small constant $\delta(n) > 0$ such that if $\alpha_M \geq 1 - \delta(n)$, then M is diffeomorphic to R^n . Another result which supports stongly Petersen's conjecture has been obtained by do Carmo and the author recently in [CX].

In the present paper, we study complete manifolds with nonnegative Ricci curvature and large volume growth. Let M be a complete manifold and $p \in M$ be fixed; we say that $K_p^{\min} \geq c$ if for any minimal geodesic γ issuing from p all sectional curvatures of the planes which are tangent to γ are greater than or equal to c . This notion was first introduced by Klingenberg [K].

Theorem 1.1. *Let (M, g) be a complete Riemannian n -manifold with Ricci curvature $\text{Ric}_M \geq 0$, $\alpha_M > 0$. Suppose that $K_p^{\min} \geq -C$ for some point $p \in M$ and some positive constant C . If for all $r > 0$, we have*

$$\frac{\text{vol}[B(p,r)]}{\omega_n r^n} < \left\{ 1 + 2^{-n} \left(\frac{1}{8\sqrt{C}r} \log \left(\frac{2}{1 + e^{-2\sqrt{C}r}} \right) \right)^{n-1} \right\} \alpha_M, \tag{1.2}$$

then M is diffeomorphic to R^n .

The following result is a generalization of Shen's theorem mentioned above.

Theorem 1.2. *Let (M, g) be a complete Riemannian n -manifold with Ricci curvature $\text{Ric}_M \geq 0$, $\alpha_M > 0$. Suppose that $K_p^{\min} \geq -C$ for some $p \in M$ and $C > 0$. If*

$$\limsup_{r \rightarrow +\infty} \left\{ \left(\frac{\text{vol}[B(p,r)]}{\omega_n r^n} - \alpha_M \right) r^{n-1} \right\} < 2^{-n} \left(\frac{\log 2}{8\sqrt{C}} \right)^{n-1} \alpha_M, \tag{1.3}$$

then M has finite topological type.

Let (M, g) be an n -dimensional complete noncompact Riemannian manifold. Fix a point $p \in M$. For any $r > 0$, let

$$k_p(r) := \inf_{M \setminus B(p,r)} K$$

where $B(p, r)$ is the open geodesic ball around p with radius r , K denotes the sectional curvature of M , and the infimum is taken over all the sections at all points on $M \setminus B(p, r)$. It is easy to see that $k_p(r) \leq 0$ and that $k_p(r)$ is a monotone function of r .

U. Abresch [A] proved that if $\int_0^\infty r k_p(r) dr > -\infty$, then M is of finite topological type. Recently, Sha and Shen [SS] showed that a complete open Riemannian manifold M has finite topological type if $\text{Ric}_M \geq 0$, $\alpha_M > 0$ and

$$k_p(r) \geq -\frac{C}{1+r^2} \tag{1.4}$$

for some constant $C > 0$ and all $r > 0$.

In this paper we then prove the

Theorem 1.3. *Given $C > 0$, and an integer $n \geq 2$, there is a positive constant $\epsilon = \epsilon(n, C)$ such that any complete Riemannian n -manifold M with Ricci curvature $\text{Ric}_M \geq 0$, $\alpha_M > 0$, $k_p(r) \geq -\frac{C}{1+r^2}$ and*

$$\frac{\text{vol}[B(p, r)]}{\omega_n r^n} \leq (1 + \epsilon)\alpha_M \tag{1.5}$$

for some $p \in M$ and all $r > 0$ is diffeomorphic to R^n .

Now we list the following Toponogov-type comparison theorem for complete manifolds with $K_p^{\min} \geq c$ obtained by Machigashira which will be used in this paper. Let $M^2(c)$ be the complete simply connected surface of constant curvature c . Throughout this paper, all geodesics are assumed to have unit speed.

Lemma 1.1 ([M1], [M2]) *Let M be a complete Riemannian manifold and p be a point of M with $K_p^{\min} \geq c$.*

(i) *Let $\gamma_i : [0, l_i] \rightarrow M$, $i = 0, 1, 2$ be minimal geodesics with $\gamma_1(0) = \gamma_2(l_2) = p$, $\gamma_0(0) = \gamma_1(l_1)$ and $\gamma_0(l_0) = \gamma_2(0)$. Then, there exist minimal geodesics $\tilde{\gamma}_i : [0, l_i] \rightarrow M^2(c)$, $i = 0, 1, 2$ with $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(l_2)$, $\tilde{\gamma}_0(0) = \tilde{\gamma}_1(l_1)$ and $\tilde{\gamma}_0(l_0) = \tilde{\gamma}_2(0)$ which are such that*

$$L(\gamma_i) = L(\tilde{\gamma}_i) \text{ for } i = 0, 1, 2$$

and

$$\begin{aligned} \angle(-\gamma'_1(l_1), \gamma'_0(0)) &\geq \angle(-\tilde{\gamma}'_1(l_1), \tilde{\gamma}'_0(0)), \\ \angle(-\gamma'_0(l_0), \gamma'_2(0)) &\geq \angle(-\tilde{\gamma}'_0(l_0), \tilde{\gamma}'_2(0)). \end{aligned}$$

(ii) *Let $\gamma_i : [0, l_i] \rightarrow M$, $i = 1, 2$ be two minimizing geodesics starting from p . Let $\tilde{\gamma}_i : [0, l_i] \rightarrow M^2(c)$ for $i = 1, 2$ be minimizing geodesics starting from same point such that $\angle(\gamma'_1(0), \gamma'_2(0)) = \angle(\tilde{\gamma}'_1(0), \tilde{\gamma}'_2(0))$. Then $d(\gamma_1(l_1), \gamma_2(l_2)) \leq d_c(\tilde{\gamma}_1(l_1), \tilde{\gamma}_2(l_2))$, where d_c denotes the distance function in $M^2(c)$.*

2. Proof of Theorem 1.1 and Theorem 1.2

Let M be an n -dimensional Riemannian manifold and $1 \leq k \leq n - 1$. If for any point $x \in M$ and any $(k+1)$ -mutually orthogonal unit tangent vectors $e, e_1, \dots, e_k \in T_x M$, we have $\sum_{i=1}^k K(e \wedge e_i) \geq 0$, we say that the k -th Ricci curvature of M is nonnegative and denote this fact by $\text{Ric}_M^{(k)} \geq 0$. Here, $K(e \wedge e_i)$ denote the sectional curvature of the plane spanned by e and $e_i (1 \leq i \leq k)$. Notice that if $\text{Ric}_M^{(k)} \geq 0$ then $\text{Ric}_M \geq 0$.

We shall prove the following more general theorem than Theorem 1.1.

Theorem 2.1. *Let (M, g) be a complete Riemannian n -manifold with $\text{Ric}_M^{(k)} \geq 0$, $\alpha_M > 0$. Suppose that $K_p^{\min} \geq -C$ for some $C > 0$ and $p \in M$. If for all $r > 0$, we have*

$$\frac{\text{vol}[B(p, r)]}{\omega_n r^n} < \left\{ 1 + 2^{-n} \left(\frac{1}{8\sqrt{C}r} \log \left(\frac{2}{1 + e^{-2\sqrt{C}r}} \right) \right)^{\frac{kn}{k+1}} \right\} \alpha_M, \tag{2.1}$$

then M is diffeomorphic to R^n .

For a point $p \in M$; we set $d_p(x) = d(p, x)$. Notice that the distance function d_p is not a smooth function (on the cut locus of p). Hence the critical points of d_p are not defined in a usual sense. The notion of critical points of d_p was introduced by Grove-Shiohama [GS].

A point $q (\neq p) \in M$ is called a critical point of d_p if there is, for any non-zero vector $v \in T_q M$, a minimal geodesic γ from q to p making an angle $\angle(v, \gamma'(0)) \leq \frac{\pi}{2}$ with v . We simply say that q is a critical point of p . It is now well-known that a complete noncompact Riemannian n -manifold M is diffeomorphic to R^n if there is a $p \in M$ such that p has no critical points other than p .

Let Σ be a closed subset of the unit tangent sphere $S_p M$ at $p \in M$. Let $B_\Sigma(p, r)$ denote the set of points $x \in B(p, r)$ such that there is a minimizing geodesic γ from p to x with $\frac{d\gamma}{dt}(0) \in \Sigma$. For $0 < r \leq \infty$, let $\Sigma_p(r)$ denote the set of unit vectors $v \in \Sigma$ such that the geodesic $\gamma(t) = \exp_p(tv)$ is minimizing on $[0, r)$. Notice that

$$\Sigma_p(r_2) \subset \Sigma_p(r_1), \quad 0 < r_1 < r_2; \quad \Sigma_p(\infty) = \bigcap_{r>0} \Sigma_p(r). \tag{2.2}$$

The following generalized Bishop-Gromov volume comparison theorem was observed in [S2].

Lemma 2.1. ([S2]) *Let (M, g) be a complete n -manifold with $\text{Ric}_M \geq 0$. Let $\Sigma \subset S_p M$ be a closed subset. Then the function $r \rightarrow \frac{\text{vol}[B_\Sigma(p, r)]}{\omega_n r^n}$ is monotone decreasing.*

Lemma 2.2. ([S2]) *Let (M, g) be a complete n -manifold with $\text{Ric}_M \geq 0$. The function*

$$r \rightarrow \frac{\text{vol}[B_{\Sigma_p(r)}(p, r)]}{\omega_n r^n}$$

is monotone decreasing. If in addition that M has large volume growth, then

$$\frac{\text{vol}[B_{\Sigma_p(r)}(p, r)]}{\omega_n r^n} \geq \alpha_M, \quad \forall r > 0. \tag{2.3}$$

Lemma 2.3. *Let (M, g) be a complete n -manifold with $\text{Ric}_M \geq 0$ and $\alpha_M > 0$. Then*

$$\frac{\text{vol}[B_{\Sigma_p(\infty)}(p, r)]}{\omega_n r^n} \geq \alpha_M, \quad \forall r > 0. \tag{2.4}$$

Proof. Observe that

$$\frac{\text{vol}[B_{\Sigma_p(r)}(p, r)]}{\omega_n r^n} = \frac{\text{vol}[B_{\Sigma_p(\infty)}(p, r)] + \text{vol}[B_{\Sigma_p(r) \setminus \Sigma_p(\infty)}(p, r)]}{\omega_n r^n}. \tag{2.5}$$

By the standard argument, we have

$$\text{vol}[B_{\Sigma_p(r) \setminus \Sigma_p(\infty)}(p, r)] \leq \frac{r^n}{n} \cdot \text{vol}(\Sigma_p(r) \setminus \Sigma_p(\infty)) \tag{2.6}$$

It follows from (2.2) that

$$\lim_{r \rightarrow \infty} \text{vol}(\Sigma_p(r) \setminus \Sigma_p(\infty)) = 0. \tag{2.7}$$

Substituting (2.6) into (2.5) and letting $r \rightarrow \infty$, one obtains by virtue of (2.7) and (2.3)

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{\text{vol}[B_{\Sigma_p(\infty)}(p, r)]}{\omega_n r^n} &\geq \lim_{r \rightarrow \infty} \frac{\text{Vol}[B_{\Sigma_p(r)}(p, r)]}{\omega_n r^n} \\ &\geq \alpha_M. \end{aligned}$$

Using Lemma 2.1, one obtains (2.4). □

Lemma 2.4. *Let (M, g) be a complete n -manifold with $\text{Ric}_M \geq 0$ and $\alpha_M > 0$. Let R_p denote the (point set) union of rays issuing from p . Then for any $r > 0$ and any $x \in \partial B(p, r)$,*

$$d(x, R_p) \leq 2\alpha_M^{-\frac{1}{n}} \left\{ \frac{\text{vol}[B(p, r)]}{\omega_n r^n} - \alpha_M \right\}^{\frac{1}{n}} r. \tag{2.9}$$

Proof. Let $s = d(x, R_p)$; then $s \leq r$ and

$$B(x, s) \cup B_{\Sigma_p(\infty)}(p, 2r) \subset B(p, 2r). \tag{2.10}$$

The left hand side of (2.10) is a disjoint union. By (1.1), we have

$$\text{vol}(B(x, s)) \geq \alpha_M \omega_n s^n.$$

From Lemma 2.1 and Lemma (2.3), one obtains

$$\begin{aligned} 2^n \text{vol}[B(p, r)] &\geq \text{vol}[B(p, 2r)] \\ &\geq \text{vol}[B(x, s)] + \text{vol}[B_{\Sigma_p(\infty)}(p, 2r)] \\ &\geq \alpha_M \omega_n s^n + \alpha_M \omega_n (2r)^n. \end{aligned} \tag{2.11}$$

thus

$$s^n \leq 2^n r^n \alpha_M^{-1} \left\{ \frac{\text{vol}[B(p, r)]}{\omega_n r^n} - \alpha_M \right\}.$$

This proves (2.9). □

Let $p, q \in M$. The *excess function* $e_{pq}(x)$ is defined by

$$e_{pq}(x) := d(p, x) + d(q, x) - d(p, q)$$

Lemma 2.5. ([AG, S1]) *Let (M, g) be a complete n -manifold with $\text{Ric}_M^{(k)} \geq 0$ for some $1 \leq k \leq n - 1$. Let $\gamma : [0, a] \rightarrow M$ be a minimal geodesic from p to q . Then for any $x \in M$,*

$$e_{pq}(x) \leq 8 \left(\frac{s^{k+1}}{r} \right)^{\frac{1}{k}}, \tag{2.12}$$

where $s = d(x, \gamma)$, $r = \min(d(p, x), d(q, x))$.

Let $\gamma : [0, \infty) \rightarrow M$ be a ray issuing from p and let $x \in M$. It is easy to see that $e_{p, \gamma(t)}(x) = d(p, x) + d(\gamma(t), x) - t$ is decreasing in t and that $e_{p, \gamma(t)}(x) \geq 0$. We define the excess function $e_{p, \gamma}$ associated to p and γ as

$$e_{p, \gamma}(x) = \lim_{t \rightarrow +\infty} e_{p, \gamma(t)}(x). \tag{2.13}$$

Then

$$e_{p, \gamma}(x) \leq e_{p, \gamma(t)}(x), \quad \forall t > 0. \tag{2.14}$$

Lemma 2.6. *Let (M, g) be a complete open Riemannian manifold with $K_p^{\min} \geq -C$ for some $C > 0$ and $p \in M$. Suppose that $x \neq p$ is a critical point of p . Then for any ray $\gamma : [0, \infty) \rightarrow M$ issuing from p*

$$e_{p, \gamma}(x) \geq \frac{1}{\sqrt{C}} \log \left(\frac{2}{1 + e^{-2\sqrt{C}d(p, x)}} \right). \tag{2.15}$$

Proof. For any $t > 0$, take a minimal geodesic $\sigma_t : [0, d(x, \gamma(t))] \rightarrow M$ from x to $\gamma(t)$. Since x is a critical point of p , there exists a minimal geodesic τ from x to p such that $\sigma'_t(0)$ and $\tau'(0)$ make an angle at most $\frac{\pi}{2}$. Applying Lemma 1.1 to the geodesic triangle $(\gamma|_{[0,t]}, \sigma_t, \tau)$, we obtain

$$\cosh(\sqrt{C}t) \leq \cosh\left(\sqrt{C}d(x, \gamma(t))\right) \cosh\left(\sqrt{C}d(p, x)\right). \tag{2.16}$$

Multiplying the above inequality by $2 \exp\left(\sqrt{C}(d(p, x) - t)\right)$ and letting $t \rightarrow +\infty$, we obtain

$$\exp\left(\sqrt{C}d(p, x)\right) \leq \exp\left(\sqrt{C}e_{p,\gamma}(x)\right) \cosh\left(\sqrt{C}d(p, x)\right). \tag{2.17}$$

Then Lemma 2.6 follows from (2.17).

Proof of Theorem 2.1. We shall prove that M contains no critical points of p (other than p) and therefore it is diffeomorphic to R^n . To do this, take an arbitrary point $x (\neq p) \in M$ and set $r = d(p, x)$. It follows from (2.1) and (2.9) that

$$d(x, R_p) < \left(\frac{1}{8\sqrt{C}} \log\left(\frac{2}{1 + e^{-2\sqrt{C}r}}\right)\right)^{\frac{k}{k+1}} \cdot r^{\frac{1}{k+1}}.$$

Thus we can find a ray $\gamma : [0, +\infty) \rightarrow M$ issuing from p and satisfying

$$s := d(x, \gamma) < \left(\frac{1}{8\sqrt{C}} \log\left(\frac{2}{1 + e^{-2\sqrt{C}r}}\right)\right)^{\frac{k}{k+1}} \cdot r^{\frac{1}{k+1}}. \tag{2.18}$$

Take $q \in \gamma$ such that $d(x, q) = d(x, \gamma)$. By (2.18), $d(x, q) < r$. Also one can easily deduce from triangle inequality that

$$\min(d(p, x), d(\gamma(t), x)) = r, \quad \forall t \geq 2r.$$

Thus $q \in \gamma((0, 2r))$ and so

$$d(x, \gamma|_{[0, 2r]}) = s.$$

Using (2.12), (2.14) and (2.18), we obtain

$$\begin{aligned} e_{p,\gamma}(x) &\leq e_{p,\gamma(2r)}(x) \\ &\leq 8 \left(\frac{s^{k+1}}{r}\right)^{\frac{1}{k}} \\ &< \frac{1}{\sqrt{C}} \log\left(\frac{2}{1 + e^{-2\sqrt{C}r}}\right). \end{aligned} \tag{2.19}$$

By (2.15) and (2.19), x is not a critical point of p . Thus M is diffeomorphic to R^n . This completes the proof of Theorem 2.1. \square

Theorem 1.2 is a consequence of the following more general result.

Theorem 2.2. *Let (M, g) be a complete Riemannian n -manifold with $\text{Ric}_M^{(k)} \geq 0$, $\alpha_M > 0$. Suppose that $K_p^{\min} \geq -C$ for some $p \in M$ and $C > 0$. If*

$$\limsup_{r \rightarrow +\infty} \left\{ \left(\frac{\text{vol}[B(p, r)]}{\omega_n r^n} - \alpha_M \right) r^{\frac{kn}{k+1}} \right\} < 2^{-n} \left(\frac{\log 2}{8\sqrt{C}} \right)^{\frac{kn}{k+1}} \cdot \alpha_M, \tag{2.20}$$

then M has finite topological type.

Proof of Theorem 2.2. By the Isotopy Lemma [C, G, GS], it suffices to show that for any $x \in M$, if $d(p, x)$ is large enough then x is not a critical point of p . Our assumption (2.20) enables us to find a small number $\epsilon > 0$ and a sufficiently large r_1 such that

$$\left(\frac{\text{vol}[B(p, r)]}{\omega_n r^n} - \alpha_M \right) r^{\frac{kn}{k+1}} < 2^{-n} \left(\frac{\log 2}{8\sqrt{C}} - \epsilon \right)^{\frac{kn}{k+1}} \alpha_M, \quad \forall r \geq r_1. \tag{2.21}$$

Since

$$\lim_{r \rightarrow +\infty} \log \left(\frac{2}{1 + e^{-2\sqrt{C}r}} \right) = \log 2,$$

there is a sufficiently large r_2 such that

$$\frac{\log \left(\frac{2}{1 + e^{-2\sqrt{C}r}} \right)}{8\sqrt{C}} > \frac{\log 2}{8\sqrt{C}} - \epsilon, \quad \forall r \geq r_2. \tag{2.22}$$

Let $r_0 = \max(r_1, r_2)$; then for any $r \geq r_0$ we have from (2.21) and (2.22) that

$$\begin{aligned} \frac{\text{vol}[B(p, r)]}{\omega_n r^n} &< \left\{ 1 + 2^{-n} \left(\frac{\log 2}{8\sqrt{C}} - \epsilon \right)^{\frac{kn}{k+1}} \right\} \cdot \alpha_M \\ &< \left\{ 1 + 2^{-n} \left(\frac{1}{8\sqrt{C}r} \log \left(\frac{2}{1 + e^{-2\sqrt{C}r}} \right) \right)^{\frac{kn}{k+1}} \right\} \cdot \alpha_M \end{aligned} \tag{2.23}$$

Now one can repeat the arguments as in the proof of Theorem 2.1 to prove that $M \setminus B(p, r_0)$ contains no critical points of p . Therefore M has finite topological type. This completes the proof of Theorem 2.2. \square

Proof of Theorem 1.3. Let $\delta = \delta(C) < \frac{1}{20}$ be a solution of the following inequality

$$\cosh^2(4\sqrt{C}\delta) - \cosh(6\sqrt{C}\delta) < 0. \quad (2.24)$$

We take our $\epsilon = \epsilon(n, C)$ in Theorem 1.3 to be

$$\epsilon = \left(\frac{\delta}{8}\right)^n \quad (2.25)$$

Take an arbitrary point $x (\neq p) \in M$ and let $r = d(p, x)$. It suffices to prove that x is not a critical point of p . Let $\gamma : [0, 2r] \rightarrow M$ be a minimizing geodesic from p to $q = \gamma(2r)$ such that $s := d(x, \gamma) = d(x, B_{\Sigma_p(\infty)}(p, 2r))$. Using the same arguments as in the proof of (2.9), we obtain

$$d(x, B_{\Sigma_p(\infty)}(p, 2r)) \leq 2\alpha_M^{-\frac{1}{n}} \left\{ \frac{\text{vol}[B(p, r)]}{\omega_n r^n} - \alpha_M \right\}^{\frac{1}{n}} \cdot r. \quad (2.26)$$

Take a minimizing geodesic σ from x to q . For any minimal geodesic σ_1 from x to p , let $\tilde{p} = \sigma_1(\delta r)$ and $\tilde{q} = \sigma(\delta r)$. Applying the Toponogov comparison theorem to the hinge $(\sigma|_{[0, \delta r]}, \sigma_1|_{[0, \delta r]})$ in $M - B_{\frac{r}{4}}(p)$, we have

$$\cosh\left(\frac{4\sqrt{C}}{r(x)}d(\tilde{p}, \tilde{q})\right) \leq \cosh^2(4\sqrt{C}\delta) - \sinh^2(4\sqrt{C}\delta) \cos \theta \quad (2.27)$$

where $\theta = \angle(\sigma'(0), \sigma_1'(0))$ be the angle of σ and σ_1 at x and we have used the fact that the sectional curvature of M satisfies $K_M \geq -\frac{4^2 C}{r^2}$ on $M - B_{\frac{r}{4}}(p)$. Let $m \in \gamma$ such that $d(x, m) = d(x, \gamma)$; it then follows from the triangle inequality that

$$\begin{aligned} d(\tilde{p}, \tilde{q}) &\geq d(p, q) - d(p, \tilde{p}) - d(q, \tilde{q}) \\ &= d(p, m) + d(q, m) - [d(p, x) - d(\tilde{p}, x)] \\ &\quad - [d(x, q) - d(x, \tilde{q})] \\ &= 2\delta r + [d(p, m) - d(p, x)] + [d(q, m) - d(q, x)] \\ &\geq 2\delta r - 2d(x, m). \end{aligned} \quad (2.28)$$

From (2.25), (2.26) and our assumption (1.5), we have

$$\begin{aligned} d(x, m) &= d(x, B_{\Sigma_p(\infty)}(p, 2r)) \\ &\leq 2\epsilon^{\frac{1}{n}} r \\ &\leq \frac{\delta r}{4}. \end{aligned} \quad (2.29)$$

Thus we have

$$d(\tilde{p}, \tilde{q}) \geq \frac{3}{2} \delta r. \quad (2.30)$$

Substituting (2.30) into (2.27) and using (2.24), we find that

$$\begin{aligned} \sinh^2(4\sqrt{C}\delta)\cos\theta &\leq \cosh^2(4\sqrt{C}\delta) - \cosh\left(\frac{4\sqrt{C}}{r(x)}d(\tilde{p}, \tilde{q})\right) \\ &\leq \cosh^2(4\sqrt{C}\delta) - \cosh(6\sqrt{C}\delta) \\ &< 0, \end{aligned} \quad (2.31)$$

or

$$\theta > \frac{\pi}{2}. \quad (2.32)$$

Hence x is not a critical point of p . Thus M is diffeomorphic to R^n . The theorem follows.

References

- [A] U. Abresch, Lower curvature bounds, Toponogov's Theorem and bounded topology I, *Ann. Sci. École Norm. Sup.* **18** (1985), 651-670.
- [AG] U. Abresch, D. Gromoll, On complete manifolds with nonnegative Ricci curvature, *J. Amer. Math. Soc.* **3** (1990), 355-374.
- [An] M. Anderson, On the topology of complete manifolds of nonnegative Ricci curvature, *Topology* **29** (1990) 41-55.
- [BC] R. L. Bishop, R. J. Crittenden, *Geometry of manifolds*, Academic Press, New York 1964.
- [CX] M. do Carmo, C. Y. Xia, Ricci curvature and the topology of open manifolds, preprint.
- [C] J. Cheeger, Critical points of distance functions and applications to geometry, *Lecture notes in Math.*, vol. 1504, Springer-Verlag, New York 1991, pp. 1-38.
- [CC] J. Cheeger, T. Colding, On the structure of spaces with Ricci curvature bounded from below, *J. Diff. Geom.* **46** (1997), 406-480.
- [G] M. Gromov, Curvature, diameter and Betti numbers, *Comment. Math. Helv.* **56** (1981), 179-195.
- [GLP] M. Gromov, J. Lafontaine, P. Pansu, *Structures métrique pour les variétés Riemanniennes*. Cédic/Fernand, Nathan, Paris 1981.
- [GS] K. Grove, K. Shiohama, A generalized sphere theorem, *Ann. Math.* **106** (1977), 201-211.
- [L] P. Li, Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature, *Ann. of Math.* **124** (1986), 1-21.
- [M1] Y. Machigashira, Manifolds with pinched radial curvature, *Proc. Amer. Math. Soc.* **118** (1993), 979-985.
- [M2] Y. Machigashira, Complete open manifolds of nonnegative radial curvature, *Pacific J. Math.* **165** (1994), 153-160.
- [P] G. Perelman, Manifolds of positive Ricci curvature with almost maximal volume, *J. Amer. Math. Soc.* **7** (1994), 299-305.
- [Pe] P. Petersen, Comparison geometry problem list, *Riemannian geometry (Waterloo, ON, 1993)*, 87-115, *Fields Inst. Monogr.*, 4, Amer. Math. Soc., Providence, RI 1996.
- [SS] J. Sha and Z. Shen, Complete manifolds with nonnegative Ricci curvature and quadratically nonnegatively curved infinity, *Amer. J. Math.* **119** (1997), 1399-1404.

- [S1] Z. Shen, On complete manifolds of nonnegative k th-Ricci curvature, *Trans. Amer. Math. Soc.* **338** (1993), 289-310.
- [S2] Z. Shen, Complete manifolds with nonnegative Ricci curvature and large volume growth, *Invent. Math.* **125** (1996), 393-404.

Changyu Xia
Departamento de Matemática-IE
Universidade de Brasília
Campus Universitário
70910-900-Brasília-DF
Brasil
e-mail: xia@ipe.mat.unb.br

(Received: August 17, 1998)