Skeins, SU(N) three-manifold invariants and TQFT

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Abstract. The skein theory associated to the HOMFLY polynomial invariant of oriented knots and links in the three-sphere is explored in order to provide the background results necessary for the creation of a Topological Quantum Field Theory. A simple local duality result in the skein theory is proved. It allows vector space dimensions in the theory to be correlated with the structure constants in a skein algebra associated to the solid torus.

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1. Introduction

There is an almost automatic way in which a Topological Quantum Field Theory (a TQFT), as outlined in [1], can be constructed from a three-manifold invariant that is generated by a skein theory. Of course the skein theory must satisfy *some* properties. This approach and the requisites of the skein theory are clearly explained in [2], where it is noted that the SU(2) skein theory is exemplary. The immediate purpose of this paper is to check that the skein theoretic formulation given by Y.Yokota [16], for the SU(N) invariants for framed links in three-manifolds, does indeed satisfy these requirements (at least at infinitely many roots of unity). However, in addition, various other aspects of SU(N) skein theory will also be developed. Note that this SU(N) theory is the skein theory that is based on the defining identity of the HOMFLY polynomial invariant (of oriented knots and links) with a specific substitution made for its variables.

A TQFT, in the dimensions under consideration, associates to every closed oriented surface Σ a vector space $V(\Sigma)$. It is required that $V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2)$ and that $V(\Sigma)$ be finite dimensional. Those results are here established for the SU(N) skein theory, the method used depending on the re-coupling properties of an anti-symmetriser $g^{(N)}$, an idempotent in a certain skein algebra. As explained in Section 3, if Σ is the boundary of an oriented three-manifold M (which can be assumed to be contained in S^3) then $V(\Sigma)$ is a certain quotient $\widetilde{\mathcal{S}}_N(M)$ of a skein space $\mathcal{S}_N(M)$ generated by framed links in M. It will be immediate that $V(\overline{\Sigma}) = V(\Sigma)^*$, where $\overline{\Sigma}$ is Σ with its orientation reversed and the asterisk denotes dualisation. The general philosophy of [2] (together with the existence of the three-manifold invariant from [16]) completes the creation of a TQFT. It extends V to become a covariant functor, from the category of oriented surfaces and oriented three-dimensional cobordisms with a p_1-structure, to the category of finite dimensional complex vector spaces and linear maps. A cobordism C from Σ_1 to Σ_2 is an oriented three-manifold with $\partial C = \overline{\Sigma}_1 \cup \Sigma_2$. If $\partial M_1 = \Sigma_1$ and $\partial M_2 = \Sigma_2$, evaluation of the SU(N) invariant [16] of the union, in $M_1 \cup C \cup \overline{M}_2$, of a framed link in M_1 and a framed link in $\overline{M}_2$ induces a map $V(\Sigma_1) \otimes V(\Sigma_2)^* \longrightarrow \mathbb{C}$. This corresponds in the natural way to a map $V(\Sigma_1) \longrightarrow V(\Sigma_2)$. This, modulo a factor from the p_1-structure on C, is the linear map $V(C)$ required in the functor's definition. The details of all this are fully explained in [2] and will not feature here. Here, however, further exploration shows how the dimension of $V(\Sigma)$ can be determined from the structure constants of a skein algebra of the solid torus $S^1 \times D^2$. That is made possible by a new local duality lemma, a skein theory result that, in a sense, allows the directions on any desired segments of a link to be reversed. Next, a version of 'fusion theory' is developed that, in particular, permits direct verification of the fact that the underlying three-manifold invariant of $S^1 \times \Sigma$ is equal to the dimension of $V(\Sigma)$. Finally, an SU(N) skein theory version of the Turaev-Viro invariants is established by means of an extension of the 'chain mail' method of J.D.Roberts [13].

Originally the quantum three-manifold invariants were propounded by E.Witten, [15]. Those corresponding to the Lie group SU(N) were substantiated in the work of V.G.Turaev and H.Wenzl [14] using the details of representation theory. An alternative proof of the existence of these invariants using skein theory was later given by Yokota [16]. Many of the results of his work are quoted here. This skein theory approach has advantages in that it stays close to the visualisable ideas of elementary knot theory and is able to employ 'obvious' geometric manoeuvres. It should be noted that the theory for $N = 2$ is very much easier than that for general integer values of N. That can be attributed in part to the fact that the representation theory of the Lie algebra $\mathfrak{sl}_2$ is self dual or to the fact that the Jones polynomial, [3] or [8], is (when viewed by means of the elegant 'bracket' of L.H.Kauffman [4]) really a theory for *unoriented* framed links. Representation theory for $\mathfrak{sl}_N$ is not self dual, and the HOMFLY polynomial insists that links be oriented. The SU(2) theory was established in [12] using representation theory and a simple skein theoretic method was given in [5] and [6] and was improved in [7].

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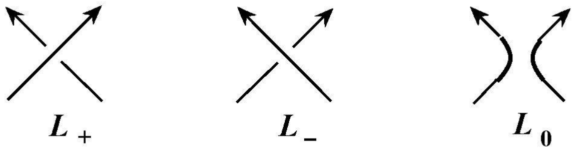

Figure 1.

2. An outline of the SU(N) skein theory of Yokota

What follows in this section is a précis of Yokota's paper [16]. Throughout, N will be a fixed integer, $N \geq 2$, and $t \in \mathbb{C}$ a primitive $2N(K+N)$ root of unity for some integer 'level' K. Let M be an oriented three-manifold together with a finite set of framed points in ∂M, each such point being oriented (that is, it has assigned to it a direction 'into' or 'out of' the manifold).

Definition 1. *The linear skein $\mathcal{S}_N(M)$ is the $\mathbb{C}$-vector space of formal linear sums of framed oriented links in M, that consist of closed curves and arcs that meet ∂M in precisely the given framed oriented points, quotiented by:*
(1) isotopy of framed links keeping boundary points fixed;
(2) $L \cup \bigcirc = [N]L$, where $[k] = (t^{kN} - t^{-kN})/(t^N - t^{-N})$;
(3) adding ± 1 to the framing of any component of L changes L to $t^{\pm(N^2-1)}L$;
(4) the skein identity $tL_+ - t^{-1}L_- = (t^N - t^{-N})L_0$ where L_+, L_- and L_0 are framed links related as in Figure 1.

Note that, when a diagram represents a framed link the framing vector is to be considered as pointing out of the paper towards the reader. Elements of $\mathcal{S}_N(M)$ will sometimes be called 'skeins in M'. The familiar theory of the HOMFLY polynomial (see [10] for example) asserts that the skein space of S^3 is one-dimensional; in what follows $\mathcal{S}_N(S^3)$ is identified with $\mathbb{C}$ by taking *the empty link* as base.

The skein space of the cube with l in-going points on its bottom face and l out-going points on its top face will be $\mathcal{S}_N(B_l^l)$. If $\xi, \eta \in \mathcal{S}_N(B_l^l)$, placing ξ above η gives a well defined product element $\xi\eta$ and so $\mathcal{S}_N(B_l^l)$ becomes an (Hecke) algebra. If $\xi \in \mathcal{S}_N(B_k^k)$ and $\eta \in \mathcal{S}_N B_l^l$ then juxtaposing cubes side by side (with the first one on the left) produces $\xi \otimes \eta \in \mathcal{S}_N(B_{k+l}^{k+l})$. Similarly a tensor product in $\mathcal{S}_N(S^1 \times D^2)$ comes by embedding two solid tori in one in a standard way. If $\xi \in \mathcal{S}_N(B_l^l)$ then $\widehat{\xi}$ denotes the element of $\mathcal{S}_N(S^1 \times D^2)$ obtained by placing the cube containing ξ in the solid torus in a standard way and joining the l points at the top of the cube to those at the bottom of the cube by l standard embedded arcs each encircling the solid torus in a positive direction. The association $\xi \mapsto \widehat{\xi}$ induces a linear map $\mathcal{S}_N(B_l^l) \to \mathcal{S}_N(S^1 \times D^2)$. Another linear map $\chi : \mathcal{S}_N(S^1 \times D^2) \to \mathbb{C}$ comes from taking an embedding, preserving orientation and framing, of $S^1 \times D^2$ onto a neighbourhood of a zero-framed unknot. Any embedding of one oriented three-manifold in another induces a linear map between their corresponding skein spaces. Suppose the disjoint union of n solid tori is embedded in M with the cores of the solid tori, with product framing, embedding on to a framed link L. If $\alpha_1, \alpha_2, \dots \alpha_n$ are elements of $\mathcal{S}_N(S^1 \times D^2)$, the image in $\mathcal{S}_N(M)$ of this n-tuple under the induced linear map will sometimes be referred to as 'the framed link L with its components *decorated* by the α_i'.

Suppose that σ_i is the usual i-th generator of the l-string braid group which provides diagrams for elements of $\mathcal{S}_N(B_l^l)$. Regarding σ_i as an element of $\mathcal{S}_N(B_l^l)$, it is clear that $\{\sigma_i^{\pm 1} : 1 \le i \le l-1\}$ is a set of generators for $\mathcal{S}_N(B_l^l)$ as an algebra. Yokota [16] focusses attention on two special elements of $\mathcal{S}_N(B_l^l)$. Both are idempotents. The first $f^{(l)}$, the symmetriser, has the property that for each i,

$$\sigma_i^{\pm 1} f^{(l)} = t^{\pm(N-1)} f^{(l)} = f^{(l)} \sigma_i^{\pm 1},$$

and the anti-symmetriser $g^{(l)}$ is such that

$$\sigma_i^{\pm 1} g^{(l)} = -t^{\mp(N+1)} g^{(l)} = g^{(l)} \sigma_i^{\pm 1}.$$

These are combined together to form further idempotents corresponding to Young diagrams. A Young diagram λ is a way of partitioning a finite set of $|\lambda|$ elements, represented by little squares, in two related ways; the sets of the partitions are the rows and columns of a diagram of little squares as on the right of Figure 2. The Young diagram λ is specified by a sequence $[l_{N-1}, l_{N-2}, \dots, l_2, l_1]$ of non-negative integers where there are $l_{N-1} + l_{N-2} + \cdots + l_i$ squares in the ith row (the top row being the first); there are $N-1$ rows though the final rows may be empty. The idempotent $e_\lambda \in \mathcal{S}_N(B_{|\lambda|}^{|\lambda|})$ is a scalar multiple of the element formed in the following way. Take an anti-symmetriser of the size of each column of the diagram and place them side by side to form

$$(g^{(N-1)})^{\otimes l_{N-1}} \otimes (g^{(N-2)})^{\otimes l_{N-2}} \otimes \cdots \otimes (g^{(1)})^{\otimes l_1} \ ,$$

connect this to the element

$$f^{(l_{N-1})} \otimes f^{(l_{N-1}+l_{N-2})} \otimes \cdots \otimes f^{(l_{N-1}+l_{N-2}+\dots l_1)}$$

formed by juxtaposing symmetrisers with sizes the rows of the diagram, and connect that to a second tensor product of anti-symmetrisers, identical to the one first mentioned, in the way shown in Figure 2 (where the g-idempotents are black rectangles and the f-idempotents are white ones).

The complex number Δ_λ is defined by

$$\Delta_\lambda = \chi(\widehat{e_\lambda}).$$

Figure 2.

In [16] it is shown that

$$\Delta_\lambda = \prod_{n=1}^{N-1} \prod_{i=1}^{n} [l_n + l_{n-1} + \cdots + l_i + n - i + 1]/[n - i + 1] \ .$$

Now, for K a fixed positive integer the set of Young diagrams $\Gamma_{N,K}$ is defined by

$$\Gamma_{N,K} = \{\lambda : \sum_{i=1}^{N-1} l_i \leq K\}$$

and the element $\Omega_K \in \mathcal{S}_N(S^1 \times D^2)$ is defined by

$$\Omega_K = \sum_{\lambda \in \Gamma_{N,K}} \Delta_\lambda \widehat{e}_\lambda .$$

Recall that t is a primitive $2N(N+K)$ root of unity. The main technical result of Yokota [16] is this: Suppose Ω_K occurs as the decoration on any component C of a framed link in S^3, then, if that link is changed by pushing (according to the fashion of a Kirby K2 move) any other component across C (retaining the Ω_K decoration on C), the evaluation of the decorated link in $\mathcal{S}_N(S^3)$ is not altered. Then, in the usual way, an $\mathrm{SU}_q(N)$ invariant for an oriented three-manifold M is constructed as is stated in the next result (from [16]).

Theorem 2. *Suppose the oriented three-manifold M is obtained by surgery on a framed link L in S^3. A well-defined invariant $\mathcal{I}_{N,K}(M)$ is defined by*

$$\mathcal{I}_{N,K}(M) = \theta \langle \theta \Omega_K \rangle_{U_-}^{\sigma} \langle \theta \Omega_K, \theta \Omega_K, \ldots, \theta \Omega_K \rangle_L .$$

Here $\langle \ \ , \ \ , \ldots, \ \ \rangle_L$ is the multilinear form on $\mathcal{S}_N(S^1 \times D^2)$ obtained by decorating all the components of L with elements of $\mathcal{S}_N(S^1 \times D^2)$ and evaluating. Also, U_- is the unknot with framing -1, the signature of the linking matrix of L

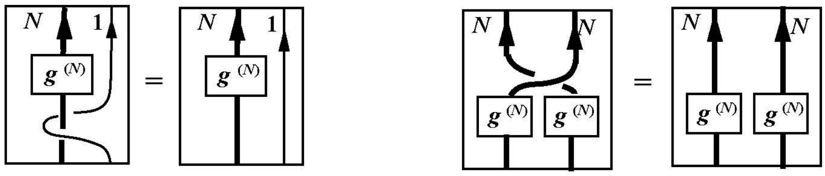

Figure 3.

is σ and $\theta \in \mathbb{C}$ is a normalising factor defined so that $\theta^{-2} = \langle \Omega_K \rangle_U$ where U is the zero-framed unknot. The main property of Ω_K mentioned above implies that

$$\langle \Omega_K \rangle_{U_+} \langle \Omega_K \rangle_{U_-} = \langle \Omega_K \rangle_U = \sum_{\lambda \in \Gamma_{N,K}} \Delta_\lambda^2$$

and that is non-zero. Yokota observes that the orientation of a curve to be decorated with Ω_K is immaterial.

3. Recoupling and fusion

The idempotent $g^{(N)}$ does not feature in the construction of e_λ for any $\lambda \in \Gamma_{N,K}$ but it is, nevertheless of special interest. It is very weak! An easy calculation shows that $\chi(\widehat{g^{(N)}}) = 1$ and that $\chi(\widehat{g^{(N+1)}}) = 0$. Suppose that M is a three-manifold, with specified framed oriented points in ∂M, contained in S^3 and let $M' = S^3 - \mathrm{Int}(M)$. In $\partial M'$ the same framed oriented boundary points are to be specified. Then the union of a framed link of arcs and simple closed curves in M with another such link in M' gives a link of framed simple closed curves in S^3. This induces a bilinear map

$$\mathcal{S}_N(M) \times \mathcal{S}_N(M') \longrightarrow \mathcal{S}_N(S^3) = \mathbb{C}.$$

Let $\widetilde{\mathcal{S}}_N(M)$ be the quotient of $\mathcal{S}_N(M)$ by the kernel of the induced linear map from $\mathcal{S}_N(M)$ to the dual of $\mathcal{S}_N(M')$. The actual chosen embedding of M in S^3 is not very important but in what follows it may be assumed that standard emdeddings are always available. It is this quotient space $\widetilde{\mathcal{S}}_N(M)$ that is of interest. The elements of $\widetilde{\mathcal{S}}_N(M)$ are represented by skeins in M, but any such skein that contains a copy of $g^{(N+1)}$ represents zero (for the result of evaluating, in $\mathcal{S}_N(S^3)$, the union of such a skein of M with any element of $\mathcal{S}_N(M')$ will contain $\chi(\widehat{g^{(N+1)}})$ as a factor). Similarly any element of $\widetilde{\mathcal{S}}_N(M)$ represented by a skein that contains a copy of $f^{(K+1)}$ is zero, because the fact that $t^{2N(N+K)} = 1$ implies that $\chi(\widehat{f^{(K+1)}}) = 0$.

Formulae (20) and (21) of [16] imply that in $\widetilde{\mathcal{S}}_N(B_{N+1}^{N+1})$ and in $\widetilde{\mathcal{S}}_N(B_{2N}^{2N})$ respectively, there are identities as shown in Figure 3, where an integer n beside a curve indicates that it represents n parallel strings.

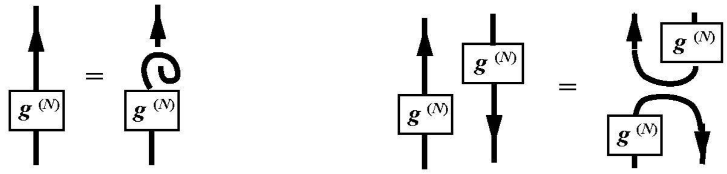

Figure 4.

Thus these are inherent relations in any $\widetilde{\mathcal{S}}_N(M)$. From them it is easy to deduce the relations of Figure 4.

These relations imply that a closed curve with any framing, decorated with $\widehat{g^{(N)}}$ may be inserted anywhere in M, to augment a skein in M, without changing the element represented in $\widetilde{\mathcal{S}}_N(M)$. That operation, together with use of the second identity of Figure 4, implies that any occurrence of $g^{(N)}$ in a skein of M may be changed, by re-routing the N strings entering $g^{(N)}$ around any path in M, without changing the element represented in $\widetilde{\mathcal{S}}_N(M)$. This can be likened to unplugging a connector of many wires (at the back of one's computer), re-routing the cable in any way and re-joining the connector. Likewise, by the second identity of Figure 4, any collection of $g^{(N)}$'s may be 'unplugged re-routed and re-joined' using any permutation of the connectors, without changing the effect in $\widetilde{\mathcal{S}}_N(M)$.

An important fusion result in the SU(N) skein theory is the following result (Proposition 2.11 of [16]).

Proposition 3. *In* $\mathcal{S}_N(B^{|\lambda|+1}_{|\lambda|+1})$,

$$e_\lambda \otimes 1 = \sum_{\kappa > \lambda} e_{[\lambda|\kappa|\lambda]} \quad .$$

Here $e_\lambda \otimes 1$ is e_λ with an additional 'straight' string to the right of it. The notation $\kappa > \lambda$ means that Young diagram κ can be obtained from Young diagram λ by adding one extra small square to λ (it may be assumed that the length of the top row of κ does not exceed K). There are then two possibilities. Either the height of the first column of κ is still less than N or it is equal to N. In the first case $e_{[\lambda|\kappa|\lambda]}$ is of the form

$$(\text{scalar})(e_\lambda \otimes 1)\sigma e_\kappa \overline{\sigma}(e_\lambda \otimes 1)$$

and in the second $e_{[\lambda|\kappa|\lambda]}$ is of the form

$$(\text{scalar})(e_\lambda \otimes 1)\sigma(g^{(N)} \otimes e_{\kappa^-})\overline{\sigma}(e_\lambda \otimes 1).$$

Here the scalars are known complex numbers, in each case σ and $\overline{\sigma}$ represent a certain braid element and its inverse, and κ^- is κ with the first column removed

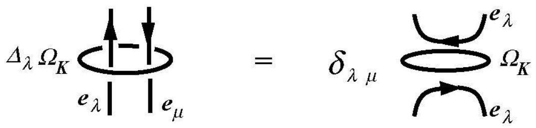

Figure 5.

(see [16]). Iteration of this result, essentially to give a consideration of $e_\lambda \otimes 1^{\otimes r}$, proves the following.

Corollary 4. *$\mathcal{S}_N(B^l_l)$ is spanned by the union of all subsets of the form*

$$\mathcal{S}_N(B^l_l)\big((g^{(N)})^{\otimes r} \otimes e_\kappa\big)\mathcal{S}_N(B^l_l)$$

where $rN + |\kappa| = l$ and $r \geq 0$. In considering the quotient of this result in $\widetilde{\mathcal{S}}_N(B^l_l)$ it may be assumed that $\kappa \in \Gamma_{N,K}$.

Another result of [16] (Lemma 4.4) is that in $\widetilde{\mathcal{S}}_N(B^{|\lambda|}_{|\lambda|})$, the element represented in $\mathcal{S}_N(B^{|\lambda|}_{|\lambda|})$ by the skein e_λ, with all its strings encircled by a zero-framed unknot decorated with Ω_K, is zero unless $\lambda = \emptyset$ (when it is, trivially, $\langle \Omega_K \rangle_U$ times the element represented by the empty diagram in $\widetilde{\mathcal{S}}_N(B^0_0)$). Now, an extension of that result was proved in [9], but only when $K + N$ is prime and $K > N$. As stated in [9], there is no real reason to suppose that that is a necessary restriction, but for the time being it must remain. The extension, to be interpretted in $\widetilde{\mathcal{S}}_N(M)$, is the result shown in Figure 5.

Theorem 5. *Suppose an element of $\widetilde{\mathcal{S}}_N(M)$ is represented by a framed oriented link that meets a two-sphere Σ, embedded in M, with zero algebraic intersection. Then, if $K + N$ is prime and $K > N$, that element may be represented by a linear sum of links each of which is disjoint from Σ.*

Proof. The strands of the framed oriented link may be isotopped so that they all cross Σ transversally near some point $P \in \Sigma$. Then, by the above corollary used twice, the link may be replaced by a linear sum of skeins each of which crosses Σ, near to P, in a copy of $(g^{(N)})^{\otimes r} \otimes e_\kappa$ oriented in one direction and $(g^{(N)})^{\otimes s} \otimes e_\tau$ in the other, for some $\kappa, \tau \in \Gamma_{N,K}$. Technically that means that these elements of the skein of a cube are introduced in cubes near P and relevant connections are then made in the remainder of $M - \Sigma$ to create the skeins to be considered. Trivially, such an element is unchanged if a small 0-framed unknot decorated with Ω_K is introduced in a small ball, disjoint from any links of the skein, and the result is multiplied by the scalar $\langle \Omega_K \rangle_U^{-1}$. (Intuitively think of the unknot as being near a point on Σ 'antipodal' to P.) That unknot may be isotopped (by stretching it over Σ) until it encircles the copies of $(g^{(N)})^{\otimes r} \otimes e_\kappa$ and $(g^{(N)})^{\otimes s} \otimes e_\tau$ which thread through it in opposite directions. The 0-framed unknot can, by the basic properties of $g^{(N)}$ (depicted in Figure 3), be allowed to pass through the copies of $g^{(N)} \otimes \cdots \otimes g^{(N)}$. Then, by the identity of Figure 5, the result represents zero unless $\kappa = \tau$. In that case the parts labelled with e_κ can be re-connected as shown in Figure 5. The result is a skein that only pierces Σ in $(g^{(N)})^{\otimes r}$ in one direction and $(g^{(N)})^{\otimes s}$ in the other. However the hypothesis that the original link meets Σ with zero algebraic intersection implies that $r = s$. Hence the copies of $g^{(N)}$ may be disconnected and rejoined (see Figure 4) so that there is then no intersection at all with Σ.

It is this theorem that is needed to prove, in the SU(N) skein-generated TQFT, that $V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2)$. The technique is carefully explained in [2]. In outline, if M is an oriented connected compact three-manifold with boundary then $V(\partial M)$ is taken to be $\widetilde{\mathcal{S}}_N(M)$. If $\partial M = \Sigma_1 \sqcup \Sigma_2$ then M can be changed to M' by surgery so that in M' the copies of Σ_1 and Σ_2 in its boundary are separated by a two-sphere Σ embedded in M'. Using the theorem $\widetilde{\mathcal{S}}_N(M')$ is generated by elements represented by skeins that do not meet Σ; these can thus be regarded as the tensor product of a skein in a manifold bounded by Σ_1 with a skein in a manifold bounded by Σ_2.

4. Local duality

Much of the remainder of this paper will use the following local duality theorem. Recall that a Young diagram λ specified by integers $[l_{N-1}, l_{N-2}, \ldots, l_2, l_1]$ has a dual Young diagram λ^* specified by $[l_1, l_2, \ldots, l_{N-2}, l_{N-1}]$. Note that $|\lambda| + |\lambda^*| = N(l_1 + l_2 + \cdots + l_{N-1})$. It was shown in [9] (Proposition 5.2) that if a skein in M contains an oriented simple closed curve decorated with $\widehat{e_\lambda}$ then the direction of the curve may be reversed and the decoration changed to $\widehat{e^*_\lambda}$ without changing the element represented in $\widetilde{\mathcal{S}}_N(M)$. A local version is as follows.

Theorem 6. *Suppose $K + N$ is prime and $K > N$. In $\widetilde{\mathcal{S}}_N(B^{|\lambda|}_{|\lambda|})$, the element represented by e_λ can also be represented by an element in the span of the elements of $\mathcal{S}_N(B^{|\lambda|}_{|\lambda|})$ depicted in* Figure 6*, where the X represents the set*

$$\mathcal{S}_N(B^{|\lambda|+|\lambda^*|}_{|\lambda|+|\lambda^*|})(g^{(N)})^{\otimes(|\lambda|+|\lambda^*|)/N}\mathcal{S}_N(B^{|\lambda|+|\lambda^*|}_{|\lambda|+|\lambda^*|}).$$

Proof. The identities shown in Figure 7 follow from that of Figure 5 and from the reversing result in [9] mentioned above.

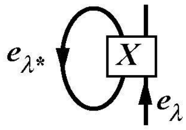

Figure 6.

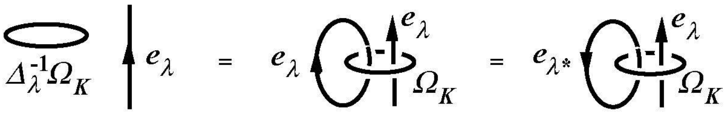

Figure 7.

In the final diagram the 0-framed unknot decorated by Ω_K is threaded by a copy of e_λ and a copy of e^*_λ both oriented in the same direction. By Corollary 4 they may be expressed as a linear sum of elements from sets of the form

$$\mathcal{S}_N(B^{|\lambda|+|\lambda^*|}_{|\lambda|+|\lambda^*|})\Big((g^{(N)})^{\otimes r}\otimes e_\kappa\Big)\mathcal{S}_N(B^{|\lambda|+|\lambda^*|}_{|\lambda|+|\lambda^*|}).$$

However, the curve decorated with Ω_K can be taken to pass through the $(g^{(N)})^{\otimes r}$ and, on encircling e_κ it gives zero unless $\kappa = \emptyset$.

In what follows, when $l < m$, $\mathcal{S}_N(B^l_l)$ is regarded as being contained in $\mathcal{S}_N(B^m_m)$ by means of the natural identification with $\mathcal{S}_N(B^l_l)\otimes 1^{\otimes(m-l)}$.

Proposition 7. *Let $\lambda_1, \lambda_2, \dots \lambda_r \in \Gamma_{N,K}$ and let $n = \sum_1^r |\lambda_i|$. Then the image in $\widetilde{\mathcal{S}}_N(B^n_n)$ of the subspace $\Big(e_{\lambda_1}\otimes e_{\lambda_2}\otimes\cdots\otimes e_{\lambda_r}\Big)\mathcal{S}_N(B^n_n)\Big(e_{\lambda_1}\otimes e_{\lambda_2}\otimes\cdots\otimes e_{\lambda_r}\Big)$ of $\mathcal{S}_N(B^n_n)$ is spanned by images of the sets*

$$\Big(e_{\lambda_1}\otimes e_{\lambda_2}\otimes\cdots\otimes e_{\lambda_r}\Big)X\Big(e_{\lambda_1}\otimes e_{\lambda_2}\otimes\cdots\otimes e_{\lambda_r}\Big)\ ,$$

where X runs through all sets of the form

$$\prod_{i=2}^r\Big(\mathcal{S}_N(B^{|\lambda_1|+\cdots+|\lambda_i|}_{|\lambda_1|+\cdots+|\lambda_i|})(g^{(N)}\otimes\cdots\otimes g^{(N)}\otimes e_{\kappa_i})\Big)$$
$$\times\prod_{j=r}^2\Big((g^{(N)}\otimes\cdots\otimes g^{(N)}\otimes e_{\mu_j})\mathcal{S}_N(B^{|\lambda_1|+\cdots+|\lambda_i|}_{|\lambda_1|+\cdots+|\lambda_i|})\Big)$$

and where $\kappa_i, \mu_j \in \Gamma_{N,K}$ and $\kappa_r = \mu_r$.

Proof. Start with $e_{\lambda_1}\otimes e_{\lambda_2}$. By Corollary 4 the image of this is in the image of the span of the sets of the form

$$(e_{\lambda_1}\otimes e_{\lambda_2})\mathcal{S}_N(B^{|\lambda_1|+|\lambda_2|}_{|\lambda_1|+|\lambda_2|})(g^{(N)}\otimes\cdots\otimes g^{(N)}\otimes e_\kappa)\mathcal{S}_N(B^{|\lambda_1|+|\lambda_2|}_{|\lambda_1|+|\lambda_2|})$$

where $\kappa \in \Gamma_{N,K}$. This can now be tensored with e_{λ_3} and the process repeated. The result of continuing such repetition is that the image in $\widetilde{\mathcal{S}}_N(B^n_n)$ of the subspace

$$\Big(e_{\lambda_1} \otimes e_{\lambda_2} \otimes \cdots \otimes e_{\lambda_r}\Big)\mathcal{S}_N(B^n_n)$$

is spanned by images of the sets of the form

$$\Big(e_{\lambda_1} \otimes e_{\lambda_2} \otimes \cdots \otimes e_{\lambda_r}\Big)\Big(\prod_{i=2}^{r}\Big(\mathcal{S}_N(B^{|\lambda_1|+\cdots+|\lambda_i|}_{|\lambda_1|+\cdots+|\lambda_i|})(g^{(N)} \otimes \cdots \otimes g^{(N)} \otimes e_{\kappa_i})\Big)\Big)\mathcal{S}_N(B^n_n).$$

Similarly, the image of the subspace

$$\mathcal{S}_N(B^n_n)\Big(e_{\lambda_1} \otimes e_{\lambda_2} \otimes \cdots \otimes e_{\lambda_r}\Big)$$

is spanned by images of the sets of the form

$$\mathcal{S}_N(B^n_n)\Big(\prod_{j=r}^{2}\Big((g^{(N)} \otimes \cdots \otimes g^{(N)} \otimes e_{\mu_j})\mathcal{S}_N(B^{|\lambda_1|+\cdots+|\lambda_j|}_{|\lambda_1|+\cdots+|\lambda_j|})\Big)\Big(e_{\lambda_1} \otimes e_{\lambda_2} \otimes \cdots \otimes e_{\lambda_r}\Big)\Big).$$

Composing these two formulations gives the required answer except for the term in the middle, namely the space

$$(g^{(N)} \otimes \cdots \otimes g^{(N)} \otimes e_{\kappa_r})(\mathcal{S}_N(B^n_n))(g^{(N)} \otimes \cdots \otimes g^{(N)} \otimes e_{\mu_r}).$$

However, by [16] Lemma 2.1 and Proposition 2.9, this represents zero unless $\kappa_r = \mu_r$. Then the $g^{(N)} \otimes \cdots \otimes g^{(N)}$ can be recoupled so that this represents the same space as $(g^{(N)} \otimes \cdots \otimes g^{(N)}) \otimes (e_{\kappa_r}\mathcal{S}_N(B^{|\kappa_r|}_{|\kappa_r|})e_{\kappa_r})$ and, by [16] Proposition 2.9, that is the space spanned by $(g^{(N)} \otimes \cdots \otimes g^{(N)}) \otimes e_{\kappa_r}$.

This result should be thought of as saying that any element of

$$\Big(e_{\lambda_1} \otimes e_{\lambda_2} \otimes \cdots \otimes e_{\lambda_r}\Big)\mathcal{S}_N(B^n_n)\Big(e_{\lambda_1} \otimes e_{\lambda_2} \otimes \cdots \otimes e_{\lambda_r}\Big)$$

can, as far as its effect on the whole theory is concerned, be replaced by a sum of elements of the form depicted (for $r = 4$) in Figure 8. Here the arrows at the top *and* bottom of the big cube are labelled $e_{\lambda_1}, e_{\lambda_2}, \ldots, e_{\lambda_r}$ and a label κ_i stands for $(g^{(N)} \otimes \cdots \otimes g^{(N)}) \otimes e_{\kappa_i}$, similarly for μ_i. The small shaded squares represent elements of the various $\mathcal{S}_N(B^l_l)$).

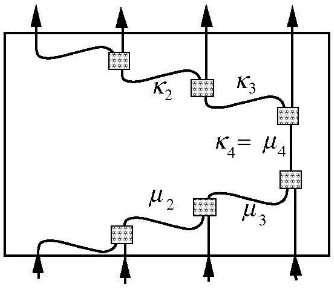

Figure 8.

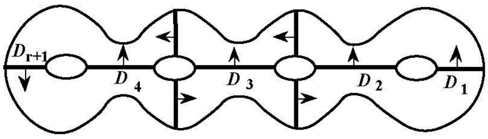

Figure 9.

5. Skeins in handlebodies

Let Σ be a closed connected orientable surface of genus r. Regard this as the boundary of a handlebody H, of genus r, which may be assumed to be embedded in S^3 in a standard way. It will be assumed that $r \geq 2$, for the case of the solid torus is relatively easy and has been described in [9]. The space $\widetilde{\mathcal{S}}_N(H)$ will now be investigated. By definition, in the TQFT, $\widetilde{\mathcal{S}}_N(H) = V(\Sigma)$. Let $D_1, D_2, \ldots, D_{3r-3}$ be a maximum set of disjoint meridian discs in H no two of which are parallel. Select a standard such set, for which cutting H along $D_1 \cup D_2 \cup \cdots \cup D_r$ gives a ball, as shown in Figure 9. Note that the $\{\partial D_i\}$ gives a 'pair of pants decomposition' for Σ. Suppose that each D_i has a standard transverse orientation, as shown. A skein in H (an element of $\mathcal{S}_N(H)$) will be said to meet D_i in $x \cup -y$, for $x \in \mathcal{S}_N(B_l^l)$ and $y \in \mathcal{S}_N(B_m^m)$ for some l and m, if it is the result of placing x in a cube embedded as a sub-cube of a neighbourhood $D_i \times I$ of D_i, with 'up' agreeing with the transverse orientation of D_i, placing y in another such cube for which 'down' agrees with the transverse orientation, and completing to form a skein in H that otherwise does not meet D_i.

Proposition 8. *Suppose $K + N$ is prime and $K > N$. A spanning set for $\widetilde{\mathcal{S}}_N(H)$ consists of elements represented by skeins that meet all $D_1, D_2, \ldots, D_r$ in $e_{\lambda_1}, e_{\lambda_2} \ldots, e_{\lambda_r}$ for some $\lambda_i \in \Gamma_{N,K}$ and meet the other D_i in elements of the form $g^{(N)} \otimes \cdots \otimes g^{(N)} \otimes e_{\kappa_i}$ for $\kappa_i \in \Gamma_{N,K}$.*

Proof. The real significance of the proposition is that the skeins cited all meet the D_i in the positive direction. $\widetilde{\mathcal{S}}_N(H)$ is certainly spanned by the classes of *all* skeins in H. The strands of such a skein can be grouped together so that, using Corollary 4, it represents the same element as a linear sum of skeins each meeting D_i, for $i = 1, 2, \ldots, r$, in elements of the form $(g^{(N)} \otimes \cdots \otimes g^{(N)} \otimes e_\lambda) \cup -(g^{(N)} \otimes \cdots \otimes g^{(N)} \otimes e_\mu)$. By using Theorem 6, that can be changed to $(g^{(N)} \otimes \cdots \otimes g^{(N)} \otimes e_\lambda \otimes e_{\mu^*}) \cup -(g^{(N)} \otimes \cdots \otimes g^{(N)})$. Then the e_λ can 'be combined with' e_{μ^*}, using Corollary 4 again, to obtain a spanning set represented by skeins that meet D_i, for $i = 1, 2, \ldots, r$, in $(g^{(N)} \otimes \cdots \otimes g^{(N)} \otimes e_{\lambda_i}) \cup -(g^{(N)} \otimes \cdots \otimes g^{(N)})$. Now, using the re-coupling properties of $g^{(N)}$, all the $g^{(N)}$ may be re-routed by way of D_{r+1} to leave skeins meeting the first r of the discs in e_{λ_i}. Now, cutting H along $D_1 \cup D_2 \cup \cdots \cup D_r$ gives a ball, and an application of Proposition 7 finishes the proof.

Note that Proposition 8 clearly gives a finite spanning set for $\widetilde{\mathcal{S}}_N(H)$, for there are only finitely many choices for the λ_i and, for each choice, the space $\mathcal{S}_N(B^{|\lambda_1|+\ldots|\lambda_r|}_{|\lambda_1|+\ldots|\lambda_r|})$, used in Proposition 7 has but finite dimension. Thus the TQFT requisite, that $V(\Sigma)$ have finite dimension, is established. However, to explore what the dimension of this space might be, a little more insight is required. With that in mind, consider spaces of the form

$$\Big((g^{(N)})^{\otimes r} \otimes e_\lambda \otimes e_\mu\Big)\mathcal{S}_N(B^l_l)\Big((g^{(N)})^{\otimes s} \otimes e_\nu\Big),$$

where $rN + |\lambda| + |\mu| = l = sN + |\nu|$. Note first that this is zero (and so can be forgotten) if $r > s$ by Lemma 2.1 of [16]. Further, if $r \leq s$ this space is, by recoupling $g^{(N)}$'s, equal to

$$(g^{(N)})^{\otimes r} \otimes \Big((e_\lambda \otimes e_\mu)\mathcal{S}_N(B^{|\lambda|+|\mu|}_{|\lambda|+|\mu|})((g^{(N)})^{\otimes(s-r)} \otimes e_\nu)\Big).$$

Similarly

$$\Big((g^{(N)})^{\otimes s} \otimes e_\nu\Big)\mathcal{S}_N(B^l_l)\Big((g^{(N)})^{\otimes r} \otimes e_\lambda \otimes e_\mu\Big)$$

is equal to

$$(g^{(N)})^{\otimes r} \otimes \Big(((g^{(N)})^{\otimes(s-r)} \otimes e_\nu)\mathcal{S}_N(B^{|\lambda|+|\mu|}_{|\lambda|+|\mu|})(e_\lambda \otimes e_\mu)\Big).$$

Now there is a natural pairing from

$$\begin{aligned}(e_\lambda \otimes e_\mu)\mathcal{S}_N(B^{|\lambda|+|\mu|}_{|\lambda|+|\mu|})&(g^{(N)} \otimes \cdots \otimes g^{(N)} \otimes e_\nu)\\ &\times\ (g^{(N)} \otimes \cdots \otimes g^{(N)} \otimes e_\nu)\mathcal{S}_N(B^{|\lambda|+|\mu|}_{|\lambda|+|\mu|})(e_\lambda \otimes e_\mu)\end{aligned}$$

to the complex numbers defined by $(x, y) \mapsto \chi(\widehat{xy})$. Let $T^{\lambda\ \mu}_\nu$ be the image of

$$(e_\lambda \otimes e_\mu)\mathcal{S}_N(B^{|\lambda|+|\mu|}_{|\lambda|+|\mu|})(g^{(N)} \otimes \cdots \otimes g^{(N)} \otimes e_\nu)$$

in the quotient space $\widetilde{\mathcal{S}}_N(B^{|\lambda|+|\mu|}_{|\lambda|+|\mu|})$ and let $T^{\nu}_{\lambda\ \mu}$ be similarly defined. The form

$$\phi : T^{\lambda\ \mu}_{\nu} \ \times\ T^{\nu}_{\lambda\ \mu} \to \mathbb{C}$$

induced by the above pairing is non-singular. Suppose $\left\{ \begin{pmatrix} \lambda\ \mu \\ \nu \end{pmatrix}_i \right\}$ and $\left\{ \begin{pmatrix} \nu \\ \lambda\ \mu \end{pmatrix}_i \right\}$, for $i = 1, 2, \dots, N_{(\lambda,\mu;\nu)}$, are bases for $T^{\lambda\ \mu}_{\nu}$ and $T^{\nu}_{\lambda\ \mu}$, respectively, so that

$$\phi\left(\begin{pmatrix} \lambda\ \mu \\ \nu \end{pmatrix}_i , \begin{pmatrix} \nu \\ \lambda\ \mu \end{pmatrix}_j \right) = \delta_{ij}.$$

Note that multiplication in $\mathcal{S}_N(B^{|\lambda|+|\mu|}_{|\lambda|+|\mu|})$ induces products

$$T^{\lambda\ \mu}_{\nu} \times T^{\nu}_{\lambda\ \mu} \to \widetilde{\mathcal{S}}_N(B^{|\lambda|+|\mu|}_{|\lambda|+|\mu|}) \quad \text{and} \quad T^{\nu}_{\lambda\ \mu} \times T^{\lambda\ \mu}_{\nu} \to \widetilde{\mathcal{S}}_N(B^{|\lambda|+|\mu|}_{|\lambda|+|\mu|}).$$

In what follows, if $x \in \mathcal{S}_N(M)$ the notation $[x]$ will be used to denote the element it represents in the quotient $\widetilde{\mathcal{S}}_N(M)$. Recall that it was shown in [9], at least when $K + N$ is prime and $K > N$, that $\widetilde{\mathcal{S}}_N(S^1 \times D^2)$ has a base $\{[\widehat{e_\lambda}] : \lambda \in \Gamma_{N,K}\}$. As has already been mentioned $\widetilde{\mathcal{S}}_N(S^1 \times D^2)$ has a tensor product (induced by placing two copies of $S^1 \times D^2$ side by side in a third copy) that turns it into an algebra.

Theorem 9. *Suppose $K + N$ is prime and $K > N$. The structure constants for the algebra $\widetilde{\mathcal{S}}_N(S^1 \times D^2)$ with respect to the base $\{[\widehat{e_\lambda}] : \lambda \in \Gamma_{N,K}\}$ are the integers $N_{\lambda,\mu;\nu}$ that are the dimensions of the spaces $T^{\lambda\ \mu}_{\nu}$ (and of the spaces $T^{\nu}_{\lambda\ \mu}$).*

Proof. By Proposition 7 the element $[e_\lambda \otimes e_\mu] \in \widetilde{\mathcal{S}}_N(B^{|\lambda|+|\mu|}_{|\lambda|+|\mu|})$ can be expressed as some linear sum of the form

$$[e_\lambda \otimes e_\mu] = \sum_{i,j,\nu} c_{i,j,\nu} \begin{pmatrix} \lambda\ \mu \\ \nu \end{pmatrix}_i \begin{pmatrix} \nu \\ \lambda\ \mu \end{pmatrix}_j$$

for some $c_{i,j,\nu} \in \mathbb{C}$, where $\nu \in \Gamma_{N,K}$. Now, by [16] Lemma 2.1 and Proposition 2.9, $\begin{pmatrix} \nu \\ \lambda\ \mu \end{pmatrix}_j \begin{pmatrix} \lambda\ \mu \\ \kappa \end{pmatrix}_k = 0$ unless $\nu = \kappa$. Also, by [16] Proposition 2.9, and the choice of the ϕ-orthonormal bases,

$$\begin{pmatrix} \nu \\ \lambda\ \mu \end{pmatrix}_j \begin{pmatrix} \lambda\ \mu \\ \nu \end{pmatrix}_k = \delta_{jk}(\Delta_\nu)^{-1}[g^{(N)} \otimes \cdots \otimes g^{(N)} \otimes e_\nu].$$

Thus, multiplying the above expression for $[e_\lambda \otimes e_\mu]$ on the right by $\begin{pmatrix} \lambda\ \mu \\ \kappa \end{pmatrix}_k$ gives

$$\begin{pmatrix} \lambda\ \mu \\ \kappa \end{pmatrix}_k = \sum_i c_{i,k,\kappa}(\Delta_\kappa)^{-1} \begin{pmatrix} \lambda\ \mu \\ \kappa \end{pmatrix}_i .$$

As all $\begin{pmatrix} \lambda\ \mu \\ \kappa \end{pmatrix}_i$ form a base of $T_\kappa^{\lambda\ \mu}$, this implies that $c_{i,k,\kappa} = \delta_{ik}\Delta_\kappa$. This means that

$$[e_\lambda \otimes e_\mu] = \sum_{i,\nu} \Delta_\nu \begin{pmatrix} \lambda\ \mu \\ \nu \end{pmatrix}_i \begin{pmatrix} \nu \\ \lambda\ \mu \end{pmatrix}_i .$$

Taking the closure of this equality in the solid torus gives

$$[\widehat{e_\lambda} \otimes \widehat{e_\mu}] = \sum_{i,\nu} \Delta_\nu d_{i,\nu} [\widehat{e_\nu}],$$

where $d_{i,\nu}$ are scalars (which exist by [16], Proposition 2.9) that satisfy

$$\begin{pmatrix} \nu \\ \lambda\ \mu \end{pmatrix}_i \begin{pmatrix} \lambda\ \mu \\ \nu \end{pmatrix}_i = d_{i,\nu}[e_\nu].$$

Taking the closure of this in S^3 gives $1 = d_{i,\nu}\Delta_\nu$. Thus

$$[\widehat{e_\lambda}] \otimes [\widehat{e_\mu}] = \sum_\nu N_{(\lambda,\mu;\nu)} [\widehat{e_\nu}]$$

which is the required formula.

Theorem 10. *Suppose $K + N$ is prime and $K > N$. Let H be a handlebody of genus r. Let $D_1, D_2, \ldots, D_{3r-3}$ be a maximum set of disjoint meridian discs (transversely oriented as above) in H, no two of which are parallel. Let $\{R_j\}$ denote the balls that form the closure of the components of $H - \cup D_i$. Then*

$$\dim \widetilde{\mathcal{S}}_N(H) = \sum_f \prod_{R_j} N_{(R_j,f)}.$$

Here the summation is over all functions $f : \{D_1, D_2, \ldots, D_{3r-3}\} \to \Gamma_{N,K}$ and $N_{(R_j,f)} = N_{(\lambda,\mu;\nu)}$ where f maps the two equally oriented discs in the boundary of R_j to λ and μ and maps the third disc to ν.

Proof. Proposition 8 gives a spanning set for $\widetilde{\mathcal{S}}_N(H)$. Each of the balls R_j has three of the D_i in its boundary, two oriented inwards and one outwards, or *vice versa*. Such an R_j can be identified with the cube that defines a $\mathcal{S}_N(B_l^l)$ with ingoing discs mapping into the bottom face and out-going discs to the top. Then, in the notation used above, it may be arranged that the any of the generators given in Proposition 8 is represented by a skein that meets each R_j in a representative of $[g^{(N)} \otimes \cdots \otimes g^{(N)}] \otimes \begin{pmatrix} \lambda\ \mu \\ \nu \end{pmatrix}_i$ or of $[g^{(N)} \otimes \cdots \otimes g^{(N)}] \otimes \begin{pmatrix} \nu \\ \lambda\ \mu \end{pmatrix}_i$. Let σ_α be such a skein, where α is an index of all the choices for λ's wherein the D_i are crossed (in the given directions) and also for these base elements in the $\{R_j\}$. Let $\widehat{H}$ be a second copy of H with meridian discs $\widehat{D_i}$, the same discs as before but with all the transverse orientations reversed. They cut $\widehat{H}$ into balls $\{\widehat{R_j}\}$. Let $\widehat{\sigma_\alpha} \in \widetilde{\mathcal{S}}_N(\widehat{H})$ be constructed from the same data as was σ_α but, when R_j contained $[g^{(N)} \otimes \cdots \otimes g^{(N)}] \otimes \begin{pmatrix} \lambda\ \mu \\ \nu \end{pmatrix}_i$, use $\rho([g^{(N)} \otimes \cdots \otimes g^{(N)}] \otimes \begin{pmatrix} \nu \\ \lambda\ \mu \end{pmatrix}_i)$ in $\widehat{R_j}$ (the same number of $g^{(N)}$'s) for $\widehat{\sigma_\alpha}$ and *vice versa.* Here ρ denotes a rotation. This rotation ρ, if considered as applied to a braid going upwards, is a π-rotation about a horizontal line, which thus produces a braid going downwards. A bilinear pairing $\psi : \widetilde{\mathcal{S}}_N(H) \times \widetilde{\mathcal{S}}_N(\widehat{H}) \to \mathbb{C}$ can be defined as follows. Place $\widehat{H}$ a little above H in S^3, take a skein in each handlebody, take zero framed unknotted curves decorated with Ω_K, one encirling *each* pair of discs $D_i \cup \widehat{D_i}$, and evaluate the resulting skein in $\mathcal{S}_N(S^3)$. Use the results depicted in Figure 3 and Figure 5, and the ϕ-orthogonality of the $\begin{pmatrix} \lambda\ \mu \\ \nu \end{pmatrix}_i$ and $\begin{pmatrix} \nu \\ \lambda\ \mu \end{pmatrix}_j$, (and the fact that $\chi(\widehat{g^{(N)}}) = 1$). It follows that $\psi(\sigma_\alpha, \widehat{\sigma_\beta}) = \delta_{\alpha\beta}(\prod_\lambda \Delta_\lambda^{-1})\langle\Omega_K\rangle_U^{3r-3}$ where the product is over all λ allocated by α to all the D_i. Thus $\{\sigma_\alpha\}$ is an independent collection and so represents a base of $\widetilde{\mathcal{S}}_N(H)$. The result now follows by simply counting this set.

Note that $N(\lambda, \mu; \nu)$ is often zero. Certainly it has been taken to be zero unless $|\lambda| + |\mu| - |\nu|$ is divisible by N. However it is also zero unless, with a proper interpretation of the role of the copies of $g^{(N)} \otimes \cdots \otimes g^{(N)}$, both $\nu \geq \lambda$ and $\nu \geq \mu$.

6. Concluding remarks

In [7] it was shown that a direct calculation for the SU(2) invariants of $S^1 \times \Sigma$ could easily be made. That can now be repeated in the case of SU(N) in the following theorem. Of course the actual result is well known to be a formal consequence of the TQFT (see [1]), so that this brief proof is but a confirmation of the whole story.

Theorem 11. *Suppose $K+N$ is prime and $K > N$. Let Σ be a closed orientable surface of genus r and let H be a handlebody of genus r. Then the invariant $\mathcal{I}_{N,K}(S^1 \times \Sigma)$ is equal to the dimension of $V(\Sigma)$, namely the dimension of $\widetilde{\mathcal{S}}_N(H)$, (which is evaluated in the previous theorem as $\sum_f \prod_{R_j} N_{(R_j,f)}$).*

Proof. The 3-manifold $S^1 \times \Sigma$ can be obtained from S^3 by surgery on a framed link L, that consists of r copies of the Borromean rings all summed together along one component, with each of its $2r+1$ unknotted components being given the zero framing. It is required to evaluate in $\mathcal{S}_N(S^3)$ the value of the skein obtained by decorating each component of this link with Ω_K (see Theorem 2). Exactly as in the analogous discussion in [7] this is equal to the evaluation of the Hopf link of two components with one component decorated with Ω_K and the other with

$$\langle \Omega_K \rangle_U^r \sum_{\lambda_i \in \Gamma_{N,K}} \left((\widehat{e_{\lambda_1}} \otimes \widehat{e_{\lambda_1^*}}) \otimes \cdots \otimes (\widehat{e_{\lambda_r}} \otimes \widehat{e_{\lambda_r^*}}) \right) .$$

This last skein should be thought of as $\widehat{e_{\lambda_1}} \otimes \cdots \otimes \widehat{e_{\lambda_r}}$, encirling the solid torus in one direction, placed beside another copy of $\widehat{e_{\lambda_1}} \otimes \cdots \otimes \widehat{e_{\lambda_r}}$ in the other direction. Then *both* copies of $\widehat{e_{\lambda_1}} \otimes \widehat{e_{\lambda_2}}$ can, by Theorem 9, be expressed in the form $\sum_\nu N_{(\lambda_1,\lambda_2;\nu)} \widehat{e_\nu}$. That can be thought of as correlating the present calculation with that given in the previous theorem for $\widetilde{\mathcal{S}}_N(H)$, when that calculation considers the two balls R_j that contain $D_1 \cup D_2$. Repetition of this argument (using Figure 5 at the final step) gives that the evaluation of the decorated link is equal to $\langle \Omega_K \rangle_U^{r+1} \dim \widetilde{\mathcal{S}}_N(H)$. However, to obtain $\mathcal{I}_{N,K}(S^1 \times \Sigma)$, this must (see Theorem 2) be multiplied by θ^{2r+2}, where $\theta^2 = \langle \Omega_K \rangle_U^{-1}$, so that the result follows at once.

Suppose that an oriented three-manifold M has a triangulation T with d_i simplexes of dimension i. It was shown in [13] that an invariant of the form of $\mathcal{I}_{N,K}(M)$ has the property that

$$|\mathcal{I}_{N,K}(M)|^2 = \theta^{2d_0+2d_2} \langle \Omega_K, \Omega_K, \ldots, \Omega_K \rangle_C$$

where C is a framed link constructed as follows. First create a link C' in M by taking a component very close to the boundary of each two-simplex and contained in that two simplex (together with its framing) and taking a component corresponding to each one-simplex that simply encircles the one-simplex linking the components in the abutting two-simplexes (give it the zero framing). This link can be considered as being contained in a regular neighbourhood of the *dual* one-skeleton. Embed this neighbourhood in any manner in S^3 and let C be the image of C'.

As already mentioned, curves decorated with Ω_K can be oriented in either direction without changing a skein evaluation. So, orient the components of C' that encircle the one-simplexes so that any two-simplex is pierced by two of these curves in one direction and one in the other. That can be achieved by ordering the vertices of the triangulation, giving the one-simplexes an orientation induced by that ordering and then orienting a curve encircling an oriented one-simplex in a positive manner with respect to the orientation on M. Now expand each Ω_K on a component encirling a one-simplex as $\sum_{\lambda \in \Gamma_{N,K}} \Delta_\lambda \widehat{e_\lambda}$. Take one term from each summation (and sum them later). Suppose that a curve decorated with Ω_K, around the edge of a two-simplex in the boundary of a three-simplex σ, is, by this process, pierced by curves labelled e_λ and e_μ pointing into σ and by one labelled e_ν pointing out of σ. In the proof of Theorem 9 it was shown that

$$[e_\lambda \otimes e_\mu] = \sum_{i,\kappa} \Delta_\kappa \begin{pmatrix} \lambda\ \mu \\ \kappa \end{pmatrix}_i \begin{pmatrix} \kappa \\ \lambda\ \mu \end{pmatrix}_i .$$

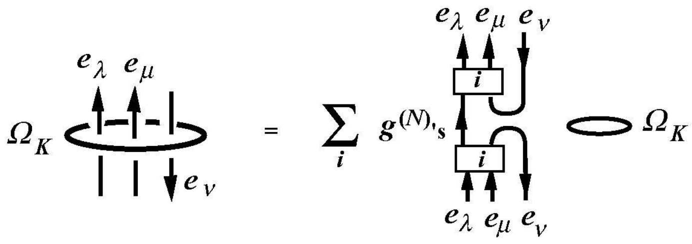

Figure 10.

Thus, using the property of Figure 5 to fuse this with the given e_ν, the situation of Figure 10 arises.

In Figure 10 the boxes labelled 'i' symbolise skeins representing $\begin{pmatrix} \lambda\,\mu \\ \nu \end{pmatrix}_i$ or $\begin{pmatrix} \nu \\ \lambda\,\mu \end{pmatrix}_i$ and the sum is over $i = 1, 2, \dots N_{(\lambda,\mu;\nu)}$.

Using this as in [13], $|\mathcal{I}_{N,K}(M)|^2$ becomes a Turaev-Viro state-sum type of invariant as in the next theorem. For the above oriented triangulation T, a *state* s is a pair of functions $s : \{\text{one-simplexes of } T\} \to \Gamma_{N,K}$ and $s : \{\text{two-simplexes of } T\} \to \mathbb{N}$ so that, for any two-simplex τ, $s(\tau) \in \{1, 2, \dots N_{s(\partial\tau)}\}$. Here $s(\partial\tau) = (\lambda, \mu; \nu)$ where the three edges of τ are mapped by s to λ, μ and ν, with the two edges oriented in the same direction around τ mapping to λ and μ. Let σ' be a three-simplex 'dual' to σ, having its vertices at the barycentres of the two-dimensional faces of σ. The edges of σ' inherit orientations from those of σ (essentially the directions of the components of C that encircle the edges of σ), with two edges oriented into each vertex of σ' and one oriented away from it or *vice versa.* Given a state s, the result of Figure 10 applied to each face of σ produces a labelling of the edges of σ' with e_{λ_i}'s, and a labelling of its vertices with various compatible (representatives of) the $\begin{pmatrix} \lambda\,\mu \\ \nu \end{pmatrix}_i$ or $\begin{pmatrix} \nu \\ \lambda\,\mu \end{pmatrix}_i$. Any $g^{(N)} \otimes \cdots \otimes g^{(N)}$ that occur from the process of Figure 10 can be decoupled and rejoined (the number of strings of any skein that cross $\partial\sigma$ must be algebraically zero). This produces a skein of the general form shown in Figure 11 (where the i_j are the $\begin{pmatrix} \lambda\,\mu \\ \nu \end{pmatrix}_i$ or $\begin{pmatrix} \nu \\ \lambda\,\mu \end{pmatrix}_i$ and the $g^{(N)}$'s that feature in the i_j's are connected together in any way). Let the evaluation of this in $\mathbb{C}$ be $E(s\ \sigma)$; it might be called a '$6j$-symbol'.

Putting this together as in [13] gives the following.

Theorem 12. *Suppose $K+N$ is prime and $K > N$. Let M be a closed orientable three-manifold with a triangulation having d_0 vertices (and with edges oriented by a vertex ordering). Then*

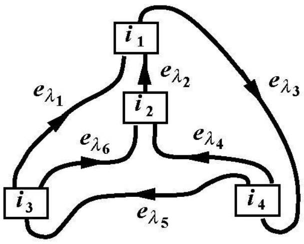

Figure 11.

$$|\mathcal{I}_{N,K}(M)|^2 = \theta^{2d_0} \sum_s (\prod_\epsilon \Delta_{s\epsilon} \prod_\sigma E(s\ \sigma)).$$

The summation is over all states s, and ϵ and σ are the edges and three-simplexes of the triangulation.

The version of the $6j$-symbols given here is as evaluations of decorated tetrahedral graphs as described above (and depicted in Figure 11). As in the case when $N = 2$ [7], these symbols can easily be interpretted as the elements of a matrix representing a change of base in (the relevant quotient of) the skein space of a ball with inputs and outputs labelled e_{λ_1}, e_{λ_2}, e_{λ_3}, e_{λ_4}, and $g^{(N)} \otimes \cdots \otimes g^{(N)}$.

The results of this paper are of a somewhat qualitative nature. To make specific calculations it would be necessary to find specific ϕ-othonornal bases for $T_\nu^{\lambda\ \mu}$ and $T^\nu_{\lambda\ \mu}$ and thus calculate the integers $N_{(\lambda,\mu;\nu)}$. Certainly some of those integers are non-zero, by [16] Proposition 2.11. For a complete numerical understanding of the theory it would be necessary to find values for the '$6j$-symbols'. For SU(2) they are explained in [7] and in [11]. The formula given in [11] for the SU(2) case is daunting enough! Perhaps, in conclusion, it should be emphasised that this theory of SU(N) invariants (which certainly *appears* to be complicated) is at least known not to be vacuous. Many non-trivial calculations have been made for SU(2). In [9] it is shown that the SU(N) invariants do distinguish apart a certain pair of manifolds (a twisted torus bundle over S^1 and the three-torus) that are not distinguished by the SU(2) theory.

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