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Autor(en): **Riedtmann, Christine / Zwara, Grzegorz**

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## On the zero set of semi-invariants for tame quivers

Christine Riedtmann and Grzegorz Zwara

**Abstract.** Let  $\mathbf{d}$  be a prehomogeneous dimension vector for a finite tame quiver  $Q$ . We show that the common zeros of all non-constant semi-invariants for the variety of representations of  $Q$  with dimension vector  $N \cdot \mathbf{d}$ , under the product of the general linear groups at all vertices, is a complete intersection for  $N \geq 3$ .

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### 1. Introduction

Let  $k$  be an algebraically closed field, and let  $Q = (Q_0, Q_1, t, h)$  be a finite quiver, i.e. a finite set  $Q_0 = \{1, \dots, n\}$  of vertices and a finite set  $Q_1$  of arrows  $\alpha : t\alpha \rightarrow h\alpha$ , where  $t\alpha$  and  $h\alpha$  denote the tail and the head of  $\alpha$ , respectively.

A representation of  $Q$  over  $k$  is a collection  $(X(i); i \in Q_0)$  of finite dimensional  $k$ -vector spaces together with a collection  $(X(\alpha) : X(t\alpha) \rightarrow X(h\alpha); \alpha \in Q_1)$  of  $k$ -linear maps. A morphism  $f : X \rightarrow Y$  between two representations is a collection  $(f(i) : X(i) \rightarrow Y(i))$  of  $k$ -linear maps such that

$$f(h\alpha) \circ X(\alpha) = Y(\alpha) \circ f(t\alpha) \quad \text{for all } \alpha \in Q_1.$$

By  $\sigma(X)$  we denote the number of pairwise non-isomorphic indecomposable direct summands occurring in a decomposition of  $X$  into indecomposables. According to the theorem of Krull–Schmidt,  $\sigma(X)$  is well-defined. The dimension vector of a representation  $X$  of  $Q$  is the vector

$$\mathbf{dim} X = (\dim X(1), \dots, \dim X(n)) \in \mathbb{N}^{Q_0}.$$

We denote the category of representations of  $Q$  by  $\text{rep}(Q)$ , and for any vector  $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^{Q_0}$

$$\text{rep}(Q, \mathbf{d}) = \prod_{\alpha \in Q_1} \text{Mat}(d_{h\alpha} \times d_{t\alpha}, k)$$

is the vector space of representations  $X$  of  $Q$  with  $X(i) = k^{d_i}$ ,  $i \in Q_0$ . The group

$$\mathrm{Gl}(\mathbf{d}) = \prod_{i=1}^n \mathrm{Gl}(d_i, k)$$

acts on  $\mathrm{rep}(Q, \mathbf{d})$  by

$$((g_1, \dots, g_n) \star X)(\alpha) = g_{h\alpha} \cdot X(\alpha) \cdot g_{t\alpha}^{-1}.$$

Note that the  $\mathrm{Gl}(\mathbf{d})$ -orbit of  $X$  consists of the representations  $Y$  in  $\mathrm{rep}(Q, \mathbf{d})$  which are isomorphic to  $X$ .

We call  $\mathbf{d}$  a prehomogeneous dimension vector if  $\mathrm{Gl}(\mathbf{d}) \star T$  is an open orbit for some  $T$  in  $\mathrm{rep}(Q, \mathbf{d})$ . Such a representation  $T$  is characterized by  $\mathrm{Ext}_Q^1(T, T) = 0$  [9]. If  $Q$  admits only finitely many indecomposable representations, or equivalently if the underlying graph of  $Q$  is a disjoint union of Dynkin diagrams of type  $\mathbb{A}$ ,  $\mathbb{D}$  or  $\mathbb{E}$  [6], every vector  $\mathbf{d}$  is prehomogeneous. Indeed, any representation is a direct sum of indecomposables and therefore  $\mathrm{rep}(Q, \mathbf{d})$  contains finitely many orbits, one of which must be open.

Let  $\mathbf{d}$  be prehomogeneous, and let  $f_1, \dots, f_s \in k[\mathrm{rep}(Q, \mathbf{d})]$  be the irreducible monic polynomials whose zeros  $Z(f_1), \dots, Z(f_s)$  are the irreducible components of codimension 1 of  $\mathrm{rep}(Q, \mathbf{d}) \setminus \mathrm{Gl}(\mathbf{d}) \star T$ , where  $\mathrm{Gl}(\mathbf{d}) \star T$  is the open orbit. It is easy to see that

$$g \cdot f_i = \chi_i(g) \cdot f_i$$

for  $g \in \mathrm{Gl}(\mathbf{d})$ , where  $\chi_i : \mathrm{Gl}(\mathbf{d}) \rightarrow k^*$  is a character. A regular function with this property is called a semi-invariant. By [11], any semi-invariant is a scalar multiple of a monomial in  $f_1, \dots, f_s$ , and  $f_1, \dots, f_s$  are algebraically independent. We denote by

$$\mathcal{Z}_{Q, \mathbf{d}} = \{X \in \mathrm{rep}(Q, \mathbf{d}); f_i(X) = 0, i = 1, \dots, s\}$$

the closed subscheme of  $\mathrm{rep}(Q, \mathbf{d})$  of common zeros of all non-constant semi-invariants. Obviously we have  $\mathrm{codim} \mathcal{Z}_{Q, \mathbf{d}} \leq s$ , and equality means that  $\mathcal{Z}_{Q, \mathbf{d}}$  is a complete intersection.

Let  $T_1, \dots, T_r$  be pairwise non-isomorphic indecomposable representations of  $Q$  such that  $\mathrm{Ext}_Q^1(T_i, T_j) = 0$  for any  $i, j \leq r$ . In [8] we showed that there is a positive integer  $N$  such that  $\mathcal{Z}_{Q, \mathbf{d}}$  is a complete intersection and irreducible for any dimension vector  $\mathbf{d} = \sum_{i=1}^r \lambda_i \mathbf{dim} T_i$  with  $\lambda_i \geq N$ ,  $i = 1, 2, \dots, r$ . Now our goal is to prove that  $N$  is quite small in case  $Q$  is tame; i.e., every connected component  $\Delta$  of  $Q$  is either a Dynkin quiver or an extended Dynkin quiver. Our methods are completely different.

Assume that  $Q$  is tame, and set

$$N(Q) = \max N(\Delta),$$

where  $\Delta$  ranges over the connected components of  $Q$  and where

$$N(\Delta) = \begin{cases} 1 & \text{if } |\Delta| = \mathbb{A}_m \text{ or } \tilde{\mathbb{A}}_m, \\ 2 & \text{if } |\Delta| = \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7 \text{ or } \mathbb{E}_8, \\ 3 & \text{if } |\Delta| = \tilde{\mathbb{D}}_m, \tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7 \text{ or } \tilde{\mathbb{E}}_8, \end{cases}$$

and  $|\Delta|$  denotes the underlying non-oriented graph of the quiver  $\Delta$ . Note that  $N(K) \leq N(Q)$  for any subquiver  $K$  of  $Q$ .

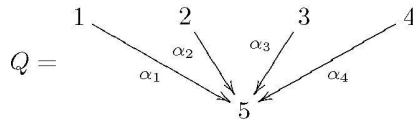
**Theorem 1.1.** *Suppose  $Q$  is tame. Let  $T_1, \dots, T_r$  be pairwise non-isomorphic indecomposable representations of  $Q$  such that  $\text{Ext}_Q^1(T_i, T_j) = 0$  for any  $i, j \leq r$ . Choose positive integers  $\lambda_1, \dots, \lambda_r$  and set  $\lambda = \min \lambda_i$ ,  $\mathbf{d} = \sum_{i=1}^r \lambda_i \mathbf{dim} T_i$ . Then  $\mathcal{Z}_{Q, \mathbf{d}}$  is*

- (i) *a complete intersection provided  $\lambda \geq N(Q)$ ,*
- (ii) *irreducible provided  $\lambda \geq N(Q) + 1$ .*

Note that the case of a Dynkin quiver of type  $\mathbb{A}_n$  has been treated by Chang and Weyman in [5].

In case  $k$  is the field  $\mathbb{C}$  of complex numbers, the fact that  $\mathcal{Z}_{Q, \mathbf{d}}$  is a complete intersection implies that  $\text{rep}(Q, \mathbf{d})$  is cofree as a representation of the subgroup  $\text{Sl}(\mathbf{d}) = \prod_{i=1}^n \text{Sl}(d_i)$  of  $\text{Gl}(\mathbf{d})$ ; i.e.,  $\mathbb{C}[\text{rep}(Q, \mathbf{d})]$  is a free module over the ring  $\mathbb{C}[\text{rep}(Q, \mathbf{d})]^{\text{Sl}(\mathbf{d})}$  of  $\text{Sl}(\mathbf{d})$ -invariant polynomials [13, §17], [8].

**Example.** Let us consider the quiver



with the dimension vector  $\mathbf{d} = \lambda \cdot \mathbf{e}$ ,  $\lambda \in \mathbb{N}$  and  $\mathbf{e} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ & & & 3 \end{pmatrix}$  as an example.

There is an indecomposable representation  $T_1$  in  $\text{rep}(Q, \mathbf{e})$ , whose orbit is open. The complement of the open orbit of  $T = T_1^\lambda$  in  $\text{rep}(Q, \mathbf{d})$  has 4 components of codimension 1, defined by

$$\det \left( X(\alpha_1) \cdots \widehat{X(\alpha_j)} \cdots X(\alpha_4) \right) = 0,$$

$j = 1, 2, 3, 4$ , where the hat means “omit  $X(\alpha_j)$ ”. Using the results developed later, we know that  $X$  belongs to  $\mathcal{Z}_{Q, \mathbf{d}}$  if and only if  $X$  either contains the simple projective  $P_5$  or else the direct sum  $\bigoplus_{j=1}^4 P_j$  of the two-dimensional projectives associated to the vertices  $1, \dots, 4$  as a direct summand. It is easy to check that

- $\mathcal{Z}_{Q, \mathbf{e}}$  is irreducible of codimension 2,
- $\mathcal{Z}_{Q, 2\mathbf{e}}$  has two irreducible components of codimension 3 and 4, respectively,
- $\mathcal{Z}_{Q, 3\mathbf{e}}$  has two irreducible components of codimension 4,
- $\mathcal{Z}_{Q, \lambda \cdot \mathbf{e}}$  is irreducible of codimension 4 for  $\lambda \geq 4$ .

## 2. Notations and preliminaries

The varieties considered in this paper are locally closed subsets of a  $k$ -vector space. If  $\mathcal{A} \subseteq \mathcal{B}$  are two such varieties and  $\mathcal{B}$  is irreducible, we denote by  $\text{codim}_{\mathcal{B}} \mathcal{A}$  the codimension of  $\mathcal{A}$  in  $\mathcal{B}$ . In case  $\mathcal{B} = \text{rep}(Q, \mathbf{d})$ , we omit the subscript  $\mathcal{B}$ .

We will assume throughout that the representation  $T = \bigoplus_{i=1}^r T_i^{\lambda_i}$  is sincere, i.e.,  $T(l) \neq 0$  for any  $l \in Q_0$ . As the full subquiver  $K$  of  $Q$  which supports  $T$  is still tame with  $N(K) \leq N(Q)$ , this is no restriction. The assumption excludes oriented cycles as subquivers of  $Q$ . Indeed, a sincere representation of an oriented cycle cannot have an open orbit.

The Euler form of  $Q$  is the  $\mathbb{Z}$ -bilinear form on  $\mathbb{Z}^{Q_0}$  defined by

$$\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} d_{t\alpha} e_{h\alpha}.$$

For  $X \in \text{rep}(Q, \mathbf{d})$ ,  $Y \in \text{rep}(Q, \mathbf{e})$  it can be computed as

$$\langle \mathbf{d}, \mathbf{e} \rangle = [X, Y] - {}^1[X, Y],$$

where

$$[X, Y] = \dim_k \text{Hom}_Q(X, Y) \quad \text{and} \quad {}^1[X, Y] = \dim_k \text{Ext}_Q^1(X, Y).$$

The quadratic form

$$q(\mathbf{d}) = \langle \mathbf{d}, \mathbf{d} \rangle$$

associated with the Euler form is the Tits form of  $Q$ . It is positive semi-definite as  $Q$  is tame and positive definite if  $Q$  does not contain extended Dynkin diagrams.

We follow Schofield [12] in order to describe the semi-invariants of  $\text{rep}(Q, \mathbf{d})$ : For a representation  $U$  of  $Q$ , the right perpendicular category  $U^\perp$  is the full subcategory of  $\text{rep}(Q)$  whose objects are

$$\{Y; [U, Y] = {}^1[U, Y] = 0\}.$$

Dually,  ${}^\perp U$  has as objects

$$\{Z; [Z, U] = {}^1[Z, U] = 0\}.$$

Note that  $U^\perp = {}^\perp(\tau U)$ , where  $\tau$  is the Auslander–Reiten translation for all non-projective indecomposable direct summands of  $U$  and  $\tau(P_l) = I_l$ , where  $P_l$  and  $I_l$  are the projective and injective indecomposable representations associated to the vertex  $l \in Q_0$ , respectively. If  ${}^1[U, U] = 0$ , the category  $U^\perp$  is equivalent to the category of representations of a quiver with  $n - \sigma(U)$  vertices.

Thus  $U^\perp$  contains  $n - r$  simple objects if  $T = \bigoplus_{i=1}^r T_i^{\lambda_i}$  is a representation of  $Q$  as in the statement of the theorem. If  $S$  is one of them, the set

$$\{X \in \text{rep}(Q, \mathbf{d}); [X, S] \neq 0\}$$

is a component of codimension 1 of the complement

$$\text{rep}(Q, \mathbf{d}) \setminus \text{Gl}(\mathbf{d}) \star T.$$

Non-isomorphic simple objects lead to distinct components, and all components of codimension 1 are obtained in this way. Thus  $\mathcal{Z}_{Q,\mathbf{d}}$  is the zero set of  $n - r$  (algebraically independent) polynomials. From now on, we will denote the underlying reduced variety of  $\mathcal{Z}_{Q,\mathbf{d}}$  by the same symbol. This will cause no confusion since we are only interested in the irreducibility and the dimension of  $\mathcal{Z}_{Q,\mathbf{d}}$ . We have the following descriptions:

$$\begin{aligned} \mathcal{Z}_{Q,\mathbf{d}} &= \{X \in \text{rep}(Q, \mathbf{d}); [X, S] \neq 0 \text{ for all simple objects } S \in T^\perp\} \\ &= \{X \in \text{rep}(Q, \mathbf{d}); [S', X] \neq 0 \text{ for all simple objects } S' \in {}^\perp T\}. \end{aligned}$$

The material presented here can be found in [12]; compare also [8]. In order to obtain part (i) of our theorem it suffices to prove  $\text{codim } \mathcal{Z}_{Q,\mathbf{d}} \geq n - r$ .

Fix a sink  $z \in Q_0$ ; i.e., a vertex  $z$  which is the head of some arrows  $\alpha_j : y_j \rightarrow z$ ,  $j = 1, \dots, s$ , but the tail of none. The vertices  $y_1, \dots, y_s$  need not be distinct. Let  $E$  be the simple projective supported at  $z$ . By  $\overline{Q}$  we denote the full subquiver of  $Q$  with  $\overline{Q}_0 = Q_0 \setminus \{z\}$  and by  $\overline{\mathbf{d}}$  the restriction of  $\mathbf{d}$  to  $\overline{Q}_0$ . Note that the orbit of the restriction  $\overline{T} = \bigoplus_{i=1}^r \overline{T}_i^{\lambda_i}$  to  $\overline{Q}$  is open in  $\text{rep}(\overline{Q}, \overline{\mathbf{d}})$ . As  $E$  is the simple projective supported at  $z$ , we have

$$E^\perp = \{X \in \text{rep}(Q); X(z) = 0\},$$

which we identify with  $\text{rep}(\overline{Q})$ . There is a short exact sequence

$$0 \rightarrow E^{d_z} \rightarrow T \rightarrow \overline{T} \rightarrow 0.$$

Considering the long exact sequence of Hom's and Ext<sup>1</sup>'s from it, we find that  $E^\perp \cap T^\perp = E^\perp \cap \overline{T}^\perp = \overline{T}^{\perp \overline{Q}}$ .

We decompose  $\mathcal{Z}_{Q,\mathbf{d}}$  as a disjoint union

$$\mathcal{Z}_{Q,\mathbf{d}} = \mathcal{Z}'_{Q,\mathbf{d}} \cup \mathcal{Z}''_{Q,\mathbf{d}},$$

where

$$\mathcal{Z}'_{Q,\mathbf{d}} = \{X \in \mathcal{Z}_{Q,\mathbf{d}}; [X, E] = 0\} \quad \text{and} \quad \mathcal{Z}''_{Q,\mathbf{d}} = \{X \in \mathcal{Z}_{Q,\mathbf{d}}; [X, E] \neq 0\}.$$

We will estimate the codimensions of  $\mathcal{Z}'_{Q,\mathbf{d}}$  and  $\mathcal{Z}''_{Q,\mathbf{d}}$  in  $\text{rep}(Q, \mathbf{d})$  separately.

Throughout the article,  $T = \bigoplus_{i=1}^r T_i^{\lambda_i}$  will denote a sincere representation of a tame quiver  $Q$ , and we set  $\lambda = \min \lambda_i \geq 1$  and  $\mathbf{dim} T = \mathbf{d}$ .

### 3. The variety $\mathcal{Z}''_{Q,\mathbf{d}}$

**Proposition 3.1.** *A representation  $X$  in  $\mathcal{Z}_{Q,\mathbf{d}}$  belongs to  $\mathcal{Z}''_{Q,\mathbf{d}}$  if and only if*

(i) *the restriction  $\overline{X}$  to  $\overline{Q}$  lies in  $\mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}}$*

and

(ii)  $\text{rank}(X(\alpha_1) \cdots X(\alpha_s)) < d_z$ .

In particular,

$$\text{codim } \mathcal{Z}''_{Q,\mathbf{d}} = \text{codim}_{\text{rep}(\overline{Q},\overline{\mathbf{d}})} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} + \max\left(0, \left(\sum_{j=1}^s d_{y_j}\right) - d_z + 1\right).$$

*Proof.* The second condition just says that  $E$  is a direct summand of  $X$ , or equivalently that  $[X, E] \neq 0$ . A representation  $X = X' \oplus E$  belongs to  $\mathcal{Z}_{Q,\mathbf{d}}$  if and only if

$$[X, S] = [X', S] + [E, S] > 0$$

for any simple object  $S \in T^\perp$ . Equivalently,

$$[X', S] > 0$$

holds for any simple representation  $S \in T^\perp$  with  $[E, S] = \dim S(z) = 0$ . These are precisely the simple objects of  $\overline{T}^{\perp\overline{Q}}$ , and moreover we have

$$[X', S] = [\overline{X}', S] = [\overline{X}, S] > 0$$

since  $S(z) = 0$ .

As for the statement about  $\text{codim } \mathcal{Z}''_{Q,\mathbf{d}}$ , observe that, in case  $d_z > \sum_{j=1}^s d_{y_j}$ , any  $d_z \times \sum_{j=1}^s d_{y_j}$ -matrix has rank less than  $d_z$ , whereas for  $d_z \leq \sum_{j=1}^s d_{y_j}$ , the subvariety

$$\mathcal{N}_{\mathbf{d}} = \left\{ A \in \text{Mat} \left( d_z \times \sum_{j=1}^s d_{y_j} \right); \text{rank } A < d_z \right\}$$

is of codimension  $\left(\sum_{j=1}^s d_{y_j}\right) - d_z + 1$ . □

**Corollary 3.2.** *Suppose that  $\lambda \geq N(Q)$  and that  $E$  is not a direct summand of  $T$ .*

(i) *We have*

$$\text{codim } \mathcal{Z}''_{Q,\mathbf{d}} - n + \sigma(T) \geq \text{codim } \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} - (n - 1) + \sigma(\overline{T}).$$

(ii) *If moreover  $d_z < \sum_{j=1}^s d_{y_j}$ , we have*

$$\text{codim } \mathcal{Z}''_{Q,\mathbf{d}} - n + \sigma(T) \geq \text{codim } \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} - (n - 1) + \sigma(\overline{T}) + \lambda - N(Q).$$

In order to prove this result, we need some information about the number  $\sigma(\overline{T})$  of pairwise non-isomorphic indecomposables occurring as direct summands of  $\overline{T}$ . We start by estimating  $\sigma(\overline{U})$  for an indecomposable representation  $U$ :

**Lemma 3.3.** *For an indecomposable representation  $U \neq E$  of  $Q$ , we have*

$$\sigma(\overline{U}) \leq 1 + N(Q) \cdot \left( \left( \sum_{j=1}^s \dim U(y_j) \right) - \dim U(z) \right).$$

*Proof.* As  $U$  is indecomposable, we may assume  $Q$  to be connected. We use the following abbreviations:

$$\dim U(z) = u, \quad \dim U(y_j) = u_j, j = 1, \dots, s, \quad u' = \left( \sum_{j=1}^s u_j \right) - u.$$

Note that  $u' \geq 0$  since  $U$  is indecomposable and  $U \neq E$ . If  $u = 0$ ,  $\bar{U} = U$  is indecomposable and  $\sigma(\bar{U}) = 1$ . In case  $u' = 0$ , the map

$$[U(\alpha_1), \dots, U(\alpha_s)] : \bigoplus_{j=1}^s U(y_j) \rightarrow U(z)$$

is an isomorphism, and again  $\bar{U}$  is indecomposable. Thus we may suppose  $u > 0$  and  $u' > 0$ .

Recall that the value of the Tits form  $q(\mathbf{dim} U)$  equals 0 or 1, as  $Q$  is tame. We compute:

$$\begin{aligned} q(\mathbf{dim} U - \mathbf{dim} E) &= q(\mathbf{dim} U) + q(\mathbf{dim} E) - \langle \mathbf{dim} U, \mathbf{dim} E \rangle - \langle \mathbf{dim} E, \mathbf{dim} U \rangle \\ &= q(\mathbf{dim} U) + q(\mathbf{dim} E) + u' - u \leq 2 + u' - u. \end{aligned}$$

As  $q$  is positive definite or positive semi-definite in case  $Q$  is a Dynkin quiver or an extended Dynkin quiver, respectively, we obtain:

$$u \leq \begin{cases} u' + 2 \leq 2u' + 1 & \text{if } Q \text{ is an extended Dynkin quiver,} \\ u' + 1 & \text{if } Q \text{ is a Dynkin quiver.} \end{cases}$$

Now clearly  $\bar{U}$  has at most  $\sum_{j=1}^s u_j$  indecomposable direct summands, and thus

$$\sigma(\bar{U}) \leq \sum_{j=1}^s u_j = u + u' \leq \begin{cases} 1 + 3u' & \text{if } Q \text{ is an extended Dynkin quiver,} \\ 1 + 2u' & \text{if } Q \text{ is a Dynkin quiver,} \end{cases}$$

which proves the lemma except in case  $|Q| = \mathbb{A}_n$  or  $|Q| = \tilde{\mathbb{A}}_{n-1}$ .

If  $|Q| = \mathbb{A}_n$ , we have  $u \leq 1$  and hence  $\sigma(\bar{U}) \leq 1 + u'$ . In case  $|Q| = \tilde{\mathbb{A}}_{n-1}$ , the number of indecomposable (possible isomorphic) direct summands in a decomposition of  $\bar{U}$  is at most  $1 + u'$ . This can be seen by inspecting the list of indecomposable representations of  $Q$ . Such representations are string or band representations, and they are described by words (non-oriented paths) in  $Q$  (see [4] for details).  $\square$

*Proof of Corollary 3.2.* We set

$$t'_i = \left( \sum_{j=1}^s \dim T_i(y_j) \right) - \dim T_i(z), \quad i = 1, \dots, r$$

and

$$t' = \sum_{i=1}^r t'_i.$$



Note that, by definition,

$$\sum_{i=1}^r \lambda_i t'_i = \left( \sum_{j=1}^s d_{y_j} \right) - d_z.$$

Our lemma implies:

$$\begin{aligned} \sigma(\bar{T}) &\leq \sum_{i=1}^r \sigma(\bar{T}_i) \leq r + N(Q) \cdot t' \leq r + \left( \sum_{i=1}^r \lambda_i t'_i \right) - (\lambda - N(Q)) \cdot t' \\ &= \sigma(T) + \left( \sum_{j=1}^s d_{y_j} \right) - d_z - (\lambda - N(Q)) \cdot t'. \end{aligned}$$

Combining this with Proposition 3.1 we find that

$$\begin{aligned} \text{codim } Z''_{\bar{Q}, \bar{\mathbf{d}}} - n + \sigma(T) &= \text{codim}_{\text{rep}(\bar{Q}, \bar{\mathbf{d}})} Z_{\bar{Q}, \bar{\mathbf{d}}} + \left( \sum_{j=1}^s d_{y_j} \right) - d_z + 1 - n + \sigma(T) \\ &\geq \text{codim}_{\text{rep}(\bar{Q}, \bar{\mathbf{d}})} Z_{\bar{Q}, \bar{\mathbf{d}}} - (n - 1) + \sigma(\bar{T}) + (\lambda - N(Q)) \cdot t'. \end{aligned}$$

As  $t'_i \geq 0$  for all  $i$ , this yields part (i) of Corollary 3.2. Part (ii) follows from the fact that  $\sum_{i=1}^r \lambda_i t'_i = \left( \sum_{j=1}^s d_{y_j} \right) - d_z > 0$  implies  $t'_i > 0$  for some  $i$  and hence  $t' > 0$ . □

#### 4. Reflection functors

We define two new quivers  $\tilde{Q}$  and  $Q'$ :  $\tilde{Q}$  is obtained from  $Q$  by adding a vertex  $z'$  and arrows  $\beta_j : z' \rightarrow y_j, j = 1, \dots, s$ . Deleting  $z$  and  $\alpha_1, \dots, \alpha_s$  in  $\tilde{Q}$  yields  $Q'$ . Note that  $Q'$  is tame as well. We denote by  $E'$  the simple injective representation of  $Q'$  supported at  $z'$ .

We consider the reflection functor

$$\mathcal{F} : \text{rep}(Q) \rightarrow \text{rep}(Q')$$

associated with  $z$ . Recall that

$$(\mathcal{F}X)(i) = \begin{cases} X(i) & i \neq z' \\ \ker \left( \bigoplus X(y_j) \xrightarrow{[X(\alpha_1), \dots, X(\alpha_s)]} X(z) \right) & i = z', \end{cases}$$

and that

$$(\mathcal{F}X)(\beta_l) : (\mathcal{F}X)(z') \rightarrow (\mathcal{F}X)(y_l) = X(y_l)$$

is the inclusion of  $(\mathcal{F}X)(z')$  into  $\bigoplus_{j=1}^s X(y_j)$  followed by the projection to  $X(y_l)$  (see [1], [6]). The functor  $\mathcal{F}$  restricts to an equivalence

$$\mathcal{F} : (\text{rep}(Q))' \rightarrow (\text{rep}(Q'))'$$

from the full subcategory  $(\text{rep}(Q))'$  of  $\text{rep}(Q)$  whose objects do not contain  $E$  as a direct summand, or equivalently have no non-trivial morphisms to  $E$ , to the full subcategory  $(\text{rep}(Q'))'$  of  $\text{rep}(Q')$  whose objects do not contain  $E'$  as a direct summand.

Suppose that  $E$  is neither a direct summand of  $T$  nor an element of  $T^\perp$ . This implies that  $[T, E] = 0$  and  ${}^1[T, E] > 0$  and thus the vector  $\mathbf{d}' \in \mathbb{Z}^{Q'_0}$ , where  $Q'_0$  denotes the set of vertices of  $Q'$ , defined by

$$d'_x = \begin{cases} d_x, & x \neq z' \\ \left( \sum_{j=1}^s d_{y_j} \right) - d_z, & x = z' \end{cases}$$

has positive entries. Indeed, we have

$$d'_{z'} = \left( \sum_{j=1}^s d_{y_j} \right) - d_z = -\langle \mathbf{d}, \mathbf{dim} E \rangle = -[T, E] + {}^1[T, E] > 0. \tag{4.1}$$

Note that in fact we have  $d'_{z'} \geq \lambda$  as  ${}^1[T_i, E] > 0$  for some  $i$  implies  ${}^1[T, E] \geq \lambda_i \geq \lambda$ . We let  $\tilde{\mathbf{d}}$  be the dimension vector for  $\tilde{Q}$  which coincides with  $\mathbf{d}$  on  $Q_0$  and with  $\mathbf{d}'$  on  $Q'_0$ .

As  $E$  is not a direct summand of  $T$ , the latter belongs to  $(\text{rep} Q)'$ . Therefore  $\mathcal{F}T$  lies in  $(\text{rep} Q')'$ , and we have  $\mathbf{dim} \mathcal{F}T = \mathbf{d}'$ ,  ${}^1[\mathcal{F}T, \mathcal{F}T] = {}^1[T, T] = 0$ , and thus  $\mathbf{d}'$  is prehomogeneous. Choose  $T'$  in the open orbit of  $\text{rep}(Q', \mathbf{d}')$ . As  $T'$  is isomorphic to  $\mathcal{F}T$ , we have  $T' = \bigoplus_{i=1}^r (T'_i)^{\lambda_i}$  with  $T'_i$  indecomposable, pairwise non-isomorphic and  ${}^1[T'_i, T'_j] = 0$  for all  $i, j$ . Moreover, we know  $T^\perp \subseteq (\text{rep} Q)'$ , as  $E$  does not belong to  $T^\perp$ , and  $(T')^\perp \subseteq (\text{rep} Q')'$ , as  $d'_{z'} = [T', E] > 0$ . We conclude that  $(T')^\perp$  is equivalent to  $\mathcal{F}(T^\perp)$ , the category of representations of a quiver with  $n - r$  vertices. Hence  $\mathcal{Z}_{Q', \mathbf{d}'}$  is given by  $n - r$  equations as well. We decompose  $\mathcal{Z}_{Q', \mathbf{d}'}$  as a disjoint union  $\mathcal{Z}_{Q', \mathbf{d}'} = \mathcal{W}'_{Q', \mathbf{d}'} \cup \mathcal{W}''_{Q', \mathbf{d}'}$ , where

$$\mathcal{W}'_{Q', \mathbf{d}'} = \{X' \in \mathcal{Z}_{Q', \mathbf{d}'}; [E', X'] = 0\}$$

and

$$\mathcal{W}''_{Q', \mathbf{d}'} = \{X' \in \mathcal{Z}_{Q', \mathbf{d}'}; [E', X'] \neq 0\}.$$

**Proposition 4.1.** *Suppose  $E$  is neither a direct summand of  $T$  nor an element of  $T^\perp$ . Then we have*

(i)  $\text{codim} \mathcal{Z}'_{Q, \mathbf{d}} = \text{codim}_{\text{rep}(Q', \mathbf{d}')} \mathcal{W}'_{Q', \mathbf{d}'}$

and

(ii)  $\mathcal{Z}'_{Q, \mathbf{d}}$  is irreducible if  $\mathcal{W}'_{Q', \mathbf{d}'}$  has this property.

*Proof.* By construction,  $X$  belongs to  $\mathcal{Z}'_{Q, \mathbf{d}}$  if and only if  $\mathcal{F}X$  is isomorphic to some  $X' \in \mathcal{W}'_{Q', \mathbf{d}'}$ , but unfortunately the functor  $\mathcal{F}$  cannot be made into a regular map from

$$\text{rep}(Q, \mathbf{d})' = \{X \in \text{rep}(Q, \mathbf{d}); [X, E] = 0\}$$

to

$$\text{rep}(Q', \mathbf{d}') = \{X' \in \text{rep}(Q', \mathbf{d}'); [E', X'] = 0\}.$$

We use the following détour (compare [7] and Section 4.2 in [3]): The set

$$\left\{ X \in \text{rep}(\tilde{Q}, \tilde{\mathbf{d}}); \sum_{j=1}^s X(\alpha_j)X(\beta_j) = 0, [X(\beta_1), \dots, X(\beta_s)]^t \text{ injective}, \right. \\ \left. [X(\alpha_1), \dots, X(\alpha_s)] \text{ surjective} \right\}$$

is a principal  $\text{Gl}(d'_z)$ -bundle over  $\text{rep}(Q, \mathbf{d})'$  and a principal  $\text{Gl}(d_z)$ -bundle over  $\text{rep}(Q', \mathbf{d}')$  via the projections  $\pi$  and  $\pi'$  deleting  $z'$  and  $z$ , respectively. Hence the claim follows from  $\pi^{-1}(\mathcal{Z}'_{Q, \mathbf{d}}) = (\pi')^{-1}(\mathcal{W}'_{Q', \mathbf{d}'})$ .  $\square$

### 5. Proof of Theorem 1.1

We proceed by induction on the number  $n$  of vertices of  $Q$ . We may assume the theorem to be true for  $\mathcal{Z}_{\tilde{Q}, \tilde{\mathbf{d}}}$ . First we treat the cases that

- (i)  $E$  is a direct summand of  $T$

and

- (ii)  $E$  belongs to  $T^\perp$ .

In both cases, we have that  $E$  is a direct summand of  $X$  for all  $X \in \mathcal{Z}_{Q, \mathbf{d}}$ ; i.e.,  $\mathcal{Z}''_{Q, \mathbf{d}} = \mathcal{Z}_{Q, \mathbf{d}}$ . Indeed, in case (i) this follows from the fact that  $\text{Hom}_Q(E, T) \neq 0$ , which is a closed condition. In case (ii),  $E$  is a simple object in  $T^\perp$ .

As any direct summand  $T_i \not\cong E$  of  $T$  belongs to  ${}^\perp E$ , we have

$$\dim T_i(z) - \sum_{j=1}^s \dim T_i(y_j) = \langle \mathbf{dim} T_i, \mathbf{dim} E \rangle = [T_i, E] - {}^1[T_i, E] = 0.$$

By Lemma 3.3,  $\overline{T}_i$  is indecomposable, and therefore

$$\sigma(\overline{T}) = \begin{cases} r - 1 & \text{in case (i),} \\ r & \text{in case (ii).} \end{cases}$$

The induction hypothesis together with Corollary 3.2 implies the first part of our theorem. We conclude from Proposition 3.1 that  $\mathcal{Z}_{Q, \mathbf{d}} \simeq \mathcal{Z}_{\tilde{Q}, \tilde{\mathbf{d}}} \times \mathcal{N}_{\mathbf{d}}$ , where

$$\mathcal{N}_{\mathbf{d}} = \{A \in \text{Mat} \left( d_z \times \sum_{j=1}^s d_{y_j} \right); \text{rank } A < d_z\}.$$

The second part follows from the fact that the set  $\mathcal{N}_{\mathbf{d}}$  is irreducible in case  $d_z \geq \sum_{j=1}^s d_{y_j}$ .

(iii) Finally, suppose that  $E$  is neither a direct summand of  $T$  nor does it belong to  $T^\perp$ , or equivalently that  $d_z < \sum_{j=1}^s d_{y_j}$ . Using Corollary 3.2 and its dual, Proposition 4.1 and remembering that the codimension of any irreducible

component of  $\mathcal{Z}_{Q,\mathbf{d}}$  is at most  $n - r$ , we see that the theorem is true for  $\mathcal{Z}_{Q,\mathbf{d}}$  if and only if it holds for  $\mathcal{Z}_{Q',\mathbf{d}'}$ .

In case either  $T$  contains a preprojective direct summand or  $T^\perp$  a preprojective representation, we may apply a series of reflection functors until we reach the situation that a simple projective either is a direct summand of  $T$  or else belongs to  $T^\perp$ , and we can reduce by (i) or (ii). This finishes the proof in case  $Q$  is of finite representation type as any indecomposable representation is preprojective.

If  $Q$  is not representation finite, we are left with the situation that no preprojective representation is a direct summand of  $T$  nor an element of  $T^\perp$ . Dually, we may assume  $T$  does not contain a preinjective direct summand either. Indeed, suppose a simple injective representation  $E'$  is a direct summand of  $T$  or belongs to  ${}^\perp T$ , a situation we will reach after a series of (inverse) reflection functors. Then apply the dual of the first or the second reduction step above; recall that  $\mathcal{Z}_{Q,\mathbf{d}}$  has a dual description as

$$\mathcal{Z}_{Q,\mathbf{d}} = \{X \in \text{rep}(Q, \mathbf{d}); [S', X] \neq 0 \text{ for all simple objects } S' \in {}^\perp T\}.$$

The following lemma finishes the proof of Theorem 1.1.

**Lemma 5.1.** *Let  $Q$  be an extended Dynkin quiver. Suppose  $T$  is a regular representation with an open orbit. Then  $T^\perp$  contains a non-zero preprojective representation.*

*Proof.* Consider a Bongartz completion  $\tilde{T}$  for  $T$  [2]; i.e., an exact sequence

$$0 \rightarrow kQ \rightarrow \tilde{T} \rightarrow \bigoplus_{i=1}^r T_i^{\nu_i} \rightarrow 0$$

for which the induced map

$$\text{Hom}_Q \left( T_l, \bigoplus_{i=1}^r T_i^{\nu_i} \right) \rightarrow \text{Ext}_Q^1(T_l, kQ)$$

is surjective for  $l = 1, \dots, r$ . There is a  $\mathbb{Z}$ -linear map  $\partial : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$ , called defect, such that any indecomposable representation  $Y$  of  $Q$  is preprojective, regular and preinjective if and only if  $\partial(\mathbf{dim} Y)$  is negative, zero and positive, respectively (see for instance [10]). As  $T$  is regular,  $\partial \tilde{T} = \partial kQ < 0$  and therefore  $\tilde{T}$  contains an indecomposable preprojective direct summand  $Y$ , and  $Y \in T^\perp$ . Indeed, we have  ${}^1[T, Y] = 0$  for all direct summands of  $\tilde{T}$  and  $[T, Y] = 0$  since  $Y$  is preprojective and  $T$  is regular [10, Theorem 3.6.5]. □

**Example.** Working out the following example, one can see that if  $Q$  is not tame, it may happen that both  $T$  and  $T^\perp$  belong to the set of regular representations:



As  $q(\mathbf{d}) = 1$ , there exists an irreducible  $T \in \text{rep}(Q, \mathbf{d})$  having an open orbit. The simple objects in  $T^\perp$  have dimension vectors

$$\begin{array}{cccccccc} 3 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ & \end{array} \quad \text{and} \quad \begin{array}{cccccccc} 2 & 2 \\ & \end{array}$$

It is easy to check that these simple objects are regular representations of  $Q$ .

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Ch. Riedtmann  
Mathematisches Institut  
Universität Bern  
Sidlerstr. 5  
CH 3012 Bern  
Switzerland  
e-mail: christine.riedtmann@math-stat.unibe.ch

G. Zwara  
Faculty of Mathematics and Computer Science  
Nicholas Copernicus University  
Chopina 12/18  
PL–87–100 Toruń  
Poland  
e-mail: gzwara@mat.uni.torun.pl

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