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On symplectic 4-manifolds with prescribed fundamental group

Scott Baldridge and Paul Kirk*

Abstract. In this article we study the problem of minimizing $a\chi + b\sigma$ on the class of all symplectic 4-manifolds with prescribed fundamental group G (χ is the Euler characteristic, σ is the signature, and $a, b \in \mathbb{R}$), focusing on the important cases χ, $\chi + \sigma$ and $2\chi + 3\sigma$. In certain situations we can derive lower bounds for these functions and describe symplectic 4-manifolds which are minimizers. We derive an upper bound for the minimum of χ and $\chi + \sigma$ in terms of the presentation of G.

1. Introduction

Pick a finitely presented group G and let $\mathfrak{M}(G)$ denote the class of closed symplectic 4-manifolds M which have $\pi_1(M)$ isomorphic to G. The existence of a symplectic M with given fundamental group G was demonstrated by Gompf [6].

In this article we study the problem of finding minimizers in $\mathfrak{M}(G)$ where minimizing is taken with regard to the Euler characteristic χ, following the approach introduced by Hausmann and Weinberger in [8] for smooth 4-manifolds. There are two aspects to this problem. Finding lower bounds to $\chi(M)$ for $M \in \mathfrak{M}(G)$ addresses the question "How large must a symplectic manifold with fundamental group G be?" The other aspect of the problem is finding efficient and explicit constructions of symplectic manifolds with a given fundamental group.

Our main general result concerning upper bounds is Theorem 6, which states:

Theorem 6. *Let G have a presentation with g generators $x_1, \ldots, x_g$ and r relations $w_1, \ldots, w_r$. Then there exists a closed symplectic 4-manifold M with $\pi_1 M \cong G$, Euler characteristic $\chi(M) = 12(g + r + 1)$, and signature $\sigma(M) = -8(g + r + 1)$.*

We also provide a number of examples of small closed symplectic manifolds with certain fundamental groups. A successful example is the following theorem, which

generalizes to the symplectic setting the results of [11]. (See Corollary 17, for a more complete statement.)

Theorem. *Let F_g denote the closed oriented surface of genus g, and suppose that $S_g = \mathrm{Sym}^2(F_g)$, so that S_g is a closed symplectic manifold with fundamental group $\mathbb{Z}^{2g}$. If $g \equiv 0, 1,$ or 3 (mod 4), then any other closed symplectic 4-manifold N with $\pi_1(N) \cong \mathbb{Z}^{2g}$ satisfies $\chi(N) \geq \chi(S_g)$.*

The general theme of this article is to investigate the simplest symplectic 4-manifolds one fundamental group at a time, finding constructions, obstructions, and examples of minimizers of $a\chi + b\sigma$.

The problem of minimizing χ and $\chi + \sigma$ of 4-manifolds with a prescribed fundamental group arises in many contexts and has been studied explicitly in a number of interesting articles. Hausmann and Weinberger in [8] used $q(G) = \min_{\pi_1(M^4) \cong G} \chi(M)$ to establish the existence of a perfect group which can be the fundamental group of a homology sphere in dimensions greater than 4 but which is not the fundamental group of a homology 4-sphere, and to construct groups which are knot groups in dimensions greater than 4 but which are not the fundamental group of a knotted 2-sphere in S^4.

Kotschick in [13] inserted the signature into the topic by defining the invariant $p(G) = \min_{\pi_1(M^4) \cong G} \chi(M) - |\sigma(M)|$ and in [14] he carries out a systematic study of $p(G)$ and $q(G)$, including computations and estimates for $q(G)$ and $p(G)$ for various G. Moreover, Kotschick discusses the problem of defining variants of p and q by restricting to 4-manifolds with fundamental group G which admit various geometric structures, e.g. spin structures, almost complex structures, positive scalar curvature, and symplectic structures, the topic of the present article. He also investigates the question of what the possible values of $p(G)$ and $q(G)$ are for a given group G, a question that we generalize and recast in Section 3.

Other related work includes the articles of Eckmann [3] and Lück [21] who derive bounds on $p(G)$ and $q(G)$ for various G using ℓ^2-cohomology and the ℓ^2-signature theorem, as well as the articles [2], [9], [11]. The general problem of calculating $q(G)$ appears as Problem 4.59 of Kirby's problem list [10].

The article is organized as follows. In Section 2 we establish some simple bounds and describe Gompf's construction for producing a symplectic 4-manifold with a given fundamental group. The function $f = a\chi + b\sigma$ for $a, b \in \mathbb{R}$ is studied in Section 3 and some reasons are given for restricting to the cases χ and $\chi + \sigma$. In Section 4 we describe new constructions that give upper bounds for $\min \chi$ and $\min \chi + \sigma$ based upon the group presentation of G. In Section 5 we focus on examples for specific classes of groups, namely free groups, cyclic groups, and free abelian groups and describe minimizers of χ for many free abelian groups. In the last section, we speculate about when or whether there are conditions for which the minimizers of χ or $\chi + \sigma$ are unique.

2. Some bounds

The fundamental numerical invariants of a 4-manifold are its Euler characteristic χ and its signature σ. We will focus on the problem of minimizing χ and sometimes $\chi + \sigma$ over the collection of symplectic manifolds with fundamental group G. Section 3 gives partial justification for our restricting to these cases. We remind the reader of some coarse bounds on the Euler characteristic of *smooth* closed orientable 4-manifolds introduced in [8]. First recall that if G is finitely generated and M is a connected oriented 4-dimensional Poincaré complex then the second Betti number of M, $b_2(M)$, is at least as large as the second Betti number of $K(G, 1)$ (with any field coefficients). Since $b_1(M) = b_3(M) = b_1(G)$, this implies

$$2 - 2b_1(G) + b_2(G) \leq \chi(M). \tag{1}$$

By taking the double of the 2-handlebody defined by a presentation of G, one obtains a smooth manifold M with $\pi_1(M) = G$ and $\chi(M) = 2 - 2d$ where d denotes the deficiency of the presentation (i.e. the number of generators minus the number of relations). Thus one has the following bound *for smooth manifolds*, where $\text{def}(G)$ denotes the minimum of the deficiency over all presentations:

$$\min_{\pi_1 M \cong G} \chi(M) \leq 2 - 2\text{def}(G).$$

This construction does not give a symplectic manifold in general. Thus this upper bound need not hold when one minimizes over symplectic manifolds with fundamental group G. To obtain a similarly general upper bound requires an examination of the construction of symplectic manifolds with prescribed fundamental group.

In the symplectic setting, Gompf has given a construction [6] by taking appropriate symplectic sums of $F \times T^2$ with many copies of the elliptic fibration $E(1)$. By examining Gompf's argument one can formalize an upper bound.

Note that any finitely presented group is the quotient of an oriented surface group, since (for example) the free group on g generators is a quotient of the fundamental group of a genus g surface. Call a system of immersed curves in general position $\gamma_i : S^1 \rightarrow F$, $i = 1, \ldots, r$ on an orientable surface F a *geometric surface presentation of* G provided the fundamental group of the 2-complex obtained by attaching 2-cells to F along the γ_i is isomorphic to G.

Given a geometric surface presentation of G, the union of the γ_i form a graph Γ (where one allows a graph to have some isolated circle components). Gompf's construction yields the following general bound.

Theorem 1 (Gompf). *Given any geometric surface presentation for G with r curves $\gamma_1, \ldots, \gamma_r$, if the associated graph Γ has n edges, there exists a closed symplectic 4-manifold M with $\pi_1(M) \cong G$, $\chi(M) = 12(r + 2n + 1)$ and $\sigma = -8(r + 2n + 1)$. Moreover, there exists a spin symplectic 4-manifold with $\pi_1(M) \cong G$, $\chi(M) = 24(r + 2n + 1)$ and $\sigma(M) = -16(r + 2n + 1)$.* $\square$

Simple experiments show that the number n in Theorem 1 can be quite large for even simple group presentations. As an example we compute the Euler characteristic of a manifold which has $G = \mathbb{Z}^4$. In this situation, start with a genus 4 surface F with a standard collection of oriented circles

$$\alpha_1, \ \alpha_2, \ \alpha_3, \ \alpha_4, \ \beta_1, \ \beta_2, \ \beta_3, \ \beta_4$$

in F representing a symplectic basis of $H_1(F)$. The quotient $\pi_1(F)/\langle \beta_1, \ldots, \beta_4 \rangle$ is a free group generated by the α_i's. For $i = 1, \ldots, 4$, let $\gamma_i = \beta_i$. For $i = 1, 2, 3$, set $\gamma_{i+4} = [\alpha_i, \alpha_{i+1}]$ using the configuration of curves on the top of F shown in Figure 1.
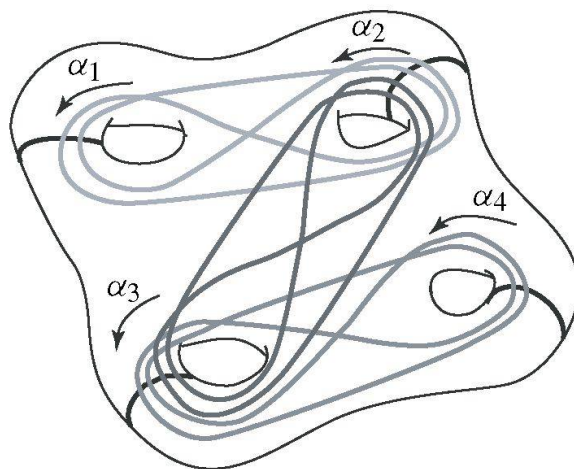
Figure 1

Finally, set $\gamma_8 = [\alpha_2, \alpha_4]$, $\gamma_9 = [\alpha_1, \alpha_4]$, $\gamma_{10} = [\alpha_1, \alpha_3]$ using the same configuration as in Figure 1, but now on the bottom of F, i.e. $\gamma_5, \gamma_6, \gamma_7$ are disjoint from $\gamma_8, \gamma_9, \gamma_{10}$. The union of the immersed curves $\gamma_1, \ldots, \gamma_{10}$ is an example of a geometric surface presentation of $\mathbb{Z}^4$. After a careful count one finds 148 edges in the graph described above. Using the theorem above one computes:

Example 2. The construction above produces a symplectic manifold M with $\pi_1(M) = \mathbb{Z}^4$ and $\chi(M) = 3,684$.

Gompf was not trying to minimize the Euler characteristic in his construction; in fact, it is clear from his writings that he knows ways to reduce this number significantly. Still, our best estimate using this construction as the starting point together with some tricks (known to us) is $\chi(M) = 516$. This is a significant reduction, no doubt, but the 4-torus T^4 has fundamental group $\mathbb{Z}^4$ and $\chi = 0$. Thus constructions like this one do not give a particularly effective upper bound for $\chi(M)$ for M symplectic, $\pi_1(M) = G$. Moreover, from the point of view of the present article the problem of expressing a bound on the number n of edges of Γ in terms of algebraic invariants of G is unwieldy in general.

We end this section by recalling two facts that are useful in increasing the lower bound of Equation (1) for symplectic manifolds. First, the symplectic form ω on a symplectic 4-manifold M has the property that $\omega \wedge \omega$ is a volume form. Thus $b^+(M)$, the dimension of the largest positive definite subspace of the intersection form (over $\mathbb{R}$) is always at least 1, and in particular, the second Betti number $b_2(M) \geq 1$. For example, this implies that if $\pi_1(M) \cong \mathbb{Z}$, then $1 \leq \chi(M)$, improving Equation (1) by one when $G = \mathbb{Z}$.

Secondly, a symplectic manifold admits an almost complex structure. This has implications on its characteristic classes. The consequence of most use to us is that $1 - b_1(M) + b^+(M)$ (the index of the ASD complex) is even. For example, if M is symplectic and $\pi_1(M) \cong \mathbb{Z}$, then $b^+(M)$ is even. Combined with the observation of the previous paragraph, we conclude that $b^+(M) \geq 2$, and hence $2 \leq \chi(M)$, improving Equation (1) by two when $G = \mathbb{Z}$.

Putting these observations together one sees that if M is symplectic with fundamental group G, then $\chi(M) + \sigma(M) = 2 - 2b_1(G) + 2b^+(M)$, and hence

$$\chi(M) + \sigma(M) \geq \begin{cases} 4 - 2b_1(G) & \text{if } b_1(G) \text{ is even,} \\ 6 - 2b_1(G) & \text{if } b_1(G) \text{ is odd.} \end{cases} \tag{2}$$

3. Minimizing $a\chi + b\sigma$ and the special points χ, $\chi + \sigma$, and $2\chi + 3\sigma$

In this section we investigate the values of a and b for which the function $a\chi + b\sigma$ has a lower bound on a suitable class of 4-manifolds with a given fundamental group (smooth, symplectic, etc.). The answers to this question naturally lead to breaking points at $a = b$, and $3a = 2b$. These are related to important invariants of symplectic 4-manifolds: $\chi + \sigma$ is 4 times the holomorphic Euler characteristic, and $2\chi + 3\sigma$ is the square of the canonical class on a symplectic manifold. The approach described in this section can be viewed as a variant of the geography problem for 4-manifolds.

We first introduce a general notion. Let $\mathfrak{M}$ denote a class of closed oriented 4-manifolds. We will be most interested in the cases, $\mathfrak{M} = \mathfrak{M}(G)$, the class of symplectic 4-manifolds with fundamental group G, $\mathfrak{M} = \mathfrak{M}^\infty(G)$, the class of

smooth manifolds with fundamental group G, and $\mathfrak{M} = \mathfrak{M}^{\min}(G)$, the subclass of $\mathfrak{M}(G)$ consisting of minimal symplectic 4-manifolds with fundamental group G (recall that a symplectic 4-manifold M is called *minimal* if it is not a blow up, i.e. $M \ncong N \# \overline{\mathbb{CP}}^2$ for N symplectic). But the following result also applies in greater generality, e.g. the class of 4-dimensional Poincaré complexes with a given fundamental group, or the class of smooth complex projective surfaces with a given fundamental group, or the class of smooth 4-manifolds with even intersection form (for which the results of [2] are relevant), or the class of almost complex 4-manifolds with given fundamental group (see [12]), or even the class of all topological oriented 4-manifolds (with no fundamental group restriction).

For $(a, b) \in \mathbb{R}^2$, define $f_{\mathfrak{M}}(a, b) \in \mathbb{R} \cup \{\pm\infty\}$ to be the infimum

$$f_{\mathfrak{M}}(a, b) = \inf_{M \in \mathfrak{M}} \{a\chi(M) + b\sigma(M)\},$$

with the understanding that $f_{\mathfrak{M}}(a, b) = \infty$ if $\mathfrak{M}$ is empty (e.g. if $\mathfrak{M}$ is the class of Kähler manifolds with fundamental group $\mathbb{Z}^3$). Define the *domain $D_{\mathfrak{M}}$ of $\mathfrak{M}$* to be the set

$$D_{\mathfrak{M}} = \{(a, b) \mid f_{\mathfrak{M}}(a, b) \neq -\infty\}.$$

Thus $D_{\mathfrak{M}}$ is the set of (a, b) so that $a\chi + b\sigma$ is bounded below on $\mathfrak{M}$. Notice that $D_{\mathfrak{M}}$ is a cone since $f_{\mathfrak{M}}(ra, rb) = r f_{\mathfrak{M}}(a, b)$ when $r \geq 0$. Furthermore, if $\mathfrak{M} \subset \mathfrak{M}'$ then $f_{\mathfrak{M}'}(a, b) \leq f_{\mathfrak{M}}(a, b)$, and hence $D_{\mathfrak{M}} \supset D_{\mathfrak{M}'}$.

Recall that a function f on a convex set S is *concave* if $f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y)$ for all $x, y \in S$ and $t \in [0, 1]$.

Theorem 3. *The domain $D_{\mathfrak{M}}$ is a convex cone and $f_{\mathfrak{M}}$ is a continuous concave function on $D_{\mathfrak{M}}$.*

Proof. The proof is simple: each $M \in \mathfrak{M}$ determines a half space $H_M \subset \mathbb{R}^3$ by

$$H_M = \{(a, b, c) \mid c \leq a\chi(M) + b\sigma(M)\}.$$

The intersection

$$I = \bigcap_{M \in \mathfrak{M}} H_M$$

is a convex set whose projection to $\mathbb{R}^2$ is $D_{\mathfrak{M}}$. Thus $D_{\mathfrak{M}}$ is convex. Furthermore, if $(a, b) \in D_{\mathfrak{M}}$, then $f_{\mathfrak{M}}(a, b)$ is the largest number c so that $(a, b, c) \in I$; this is clearly continuous and concave. $\qquad\square$

Since $D_{\mathfrak{M}}$ is a convex cone, it is either the entire plane (e.g. if $\mathfrak{M}$ contains finitely many homotopy types) or else it is a cone with angle less than or equal to π.

Interestingly, $D_{\mathfrak{M}}$ need not be closed. For example, let $\mathfrak{M} = \{M_k\}_{k=1}^{\infty}$, where

$$M_k = 2k^2 \overline{\mathbb{CP}}^2 \# (k^2 - k) S^2 \times S^2.$$

Then $\chi(M_k) = 2 + 4k^2 - 2k$ and $\sigma(M_k) = -2k^2$. Thus $a\chi + b\sigma$ has a lower bound if $2a > b$ or if $2a = b$ and $a \leq 0$. Otherwise, $a\chi + b\sigma$ is not bounded below. Thus $D_{\mathfrak{M}}$ has cone angle π and contains one of its boundary rays $(\{(-r, -2r) \mid r > 0\})$ but not the other $(\{(r, 2r) \mid r > 0\}$.

We focus now on the class $\mathfrak{M}^{\infty}(G)$ of smooth 4-manifolds with fundamental group G. Blowing up (i.e. taking the connected sum with $\overline{\mathbb{CP}}^2$) increases χ by 1 and decreases σ by 1 without changing G. Thus $a\chi + b\sigma$ is not bounded below if $a - b < 0$, and so $D_{\mathfrak{M}^{\infty}(G)}$ is contained in the half-plane $\{a \geq b\}$. Similarly, taking connected sums with $\mathbb{CP}^2$ shows that $D_{\mathfrak{M}^{\infty}(G)}$ is contained in the half-plane $\{a \geq -b\}$. Hence $D_{\mathfrak{M}^{\infty}(G)}$ lies in the cone $\{a \geq |b|\}$.

If $a \geq |b|$ then

$$a\chi(M) + b\sigma(M) = 2a(1 - b_1(G)) + (a+b)b^{+}(M) + (a-b)b^{-}(M) \geq 2a(1 - b_1(G))$$

and so $(a, b) \in D_{\mathfrak{M}^{\infty}(G)}$. Thus we have proven the following.

Proposition 4. *Fix a group G and $a \neq 0$. Then $f_{\mathfrak{M}^{\infty}(G)}$ has domain*

$$D_{\mathfrak{M}^{\infty}(G)} = \{(a, b) \mid a \geq |b|\},$$

i.e. $D_{\mathfrak{M}^{\infty}(G)}$ is the cone over the closed interval $\{1\} \times [-1, 1]$.   □

Restricting to the class of symplectic manifolds $\mathfrak{M}(G)$ everything follows as above except for one point: taking connected sum of a symplectic manifold with $\mathbb{CP}^2$ does not yield a symplectic manifold. In particular, one cannot conclude that $a\chi + b\sigma$ has no lower bound on $\mathfrak{M}(G)$ for $b < -a$. Theorem 6.3 of [6] shows that there exists symplectic manifolds with fundamental group G and arbitrarily large signature. Thus $b\sigma$ does not have a lower bound on $\mathfrak{M}(G)$ when $b < 0$.

These observations imply that the domain $D_{\mathfrak{M}(G)}$ is contained in the intersection of the half-planes $b \leq a$ and $a \geq 0$, and contains the ray $\{(r, r) \mid r \geq 0\}$ as one the two boundary edges of the cone $D_{\mathfrak{M}(G)}$. The other edge is a ray $\{(r \cos(\theta_G), r \sin(\theta_G) \mid r \geq 0\}$ for some angle θ_G in $\left[-\frac{\pi}{2}, -\frac{\pi}{4}\right]$. We were unable to determine the "critical" angle θ_G. This leads us to pose the question:

Question 1. Does the domain $D_{\mathfrak{M}(G)}$ contain any pairs (a, b) with $b < -a$?

For $G = \{e\}$, Stipsicz ([27]) has constructed simply connected symplectic 4-manifolds so that $a\chi + b\sigma$ is not bounded below when $b < -3a$, so that $\theta_{\{e\}} \geq \tan^{-1}(-3)$. Figure 2 explains the notation.

We now look at the class $\mathfrak{M}^{\min}(G)$ of minimal symplectic manifolds with fundamental group G. This time blowing up is not allowed, since by definition minimal symplectic manifolds are not blowups. Since $\mathfrak{M}^{\min}(G) \subset \mathfrak{M}^{\infty}(G)$ we know by Proposition 4 that $D_{\mathfrak{M}^{\min}(G)}$ contains the cone over the interval $\{1\} \times [-1, 1]$. The following proposition implies that $D_{\mathfrak{M}^{\min}(G)}$ is strictly larger than $D_{\mathfrak{M}(G)}$.
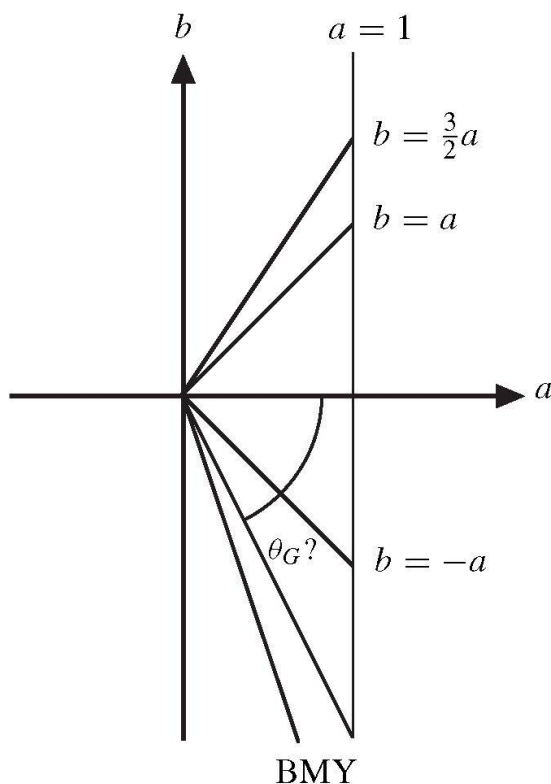
Figure 2

Proposition 5. *Fix a group G. Then $\chi + b\sigma$ has a lower bound on $\mathfrak{M}^{\min}(G)$ if $-1 \leq b \leq \frac{3}{2}$ and does not if $b > \frac{3}{2}$. In particular, $D_{\mathfrak{M}^{\min}(G)}$ (and hence $D_{\mathfrak{M}(G)}$) is contained in the half-plane $\{(a, b) \mid b \leq \frac{3}{2}a\}$, and $D_{\mathfrak{M}^{\min}(G)}$ contains the cone over the interval $\{1\} \times \left[-1, \frac{3}{2} \right]$.*

Proof. Let K be the canonical class of $M \in \mathfrak{M}^{\min}(G)$. A theorem of Liu [20] states that if $K^2 < 0$, then M is diffeomorphic to an irrationally ruled surface with fundamental group a surface group. Assume for a moment that G is not a surface group. In this case $K^2 \geq 0$ or, equivalently $2\chi(M) + 3\sigma(M) \geq 0$ for all manifolds $M \in \mathfrak{M}^{\min}(G)$. The convexity of the cone $D_{\mathfrak{M}^{\min}(G)}$ and the fact that $D_{\mathfrak{M}^{\infty}(G)} \subset D_{\mathfrak{M}^{\min}(G)}$ implies that $D_{\mathfrak{M}^{\min}(G)}$ contains the cone $\{(a, b) \mid b \leq \frac{3}{2}a$ and $a \geq -b\}$. The first part of the proposition follows from this inequality for such groups.

The case when G is a surface group is similar. (Note that in this case there are only two manifolds in $\mathfrak{M}^{\min}(G)$ up to diffeomorphism with $K^2 < 0$.)

To show that $\chi + b\sigma$ is unbounded if $b > \frac{3}{2}$, let M be a spin symplectic manifold with $\pi_1 M \cong G$ given by Gompf's construction. Then $2\chi(M) + 3\sigma(M) = 0$. By construction, M contains embedded symplectic tori with self-intersection zero and the inclusion of these tori induces the trivial morphism on fundamental groups. So one can take symplectic sums with arbitrarily many (elliptically fibered) K3 surfaces, without

changing the fundamental group. Moreover, the symplectic sums continue to be minimal by a result of Li and Stipsicz [19]. Each such sum increases χ by 24 and decreases σ by 16. Therefore $\chi + b\sigma$ can be made as small as desired when $b > \frac{3}{2}$. $\square$

The proof of Proposition 5 shows that except for surface groups (of genus $g > 1$), $f_{\mathfrak{M}^{\min}(G)}(2, 3) = 0$. For surface groups of genus $g > 1$, the only minimal symplectic manifolds with $f_{\mathfrak{M}^{\min}(G)}(2, 3) < 0$ are diffeomorphic to irrational ruled surfaces, in which case it is known that $f_{\mathfrak{M}^{\min}(G)}(2, 3) = 4(2 - 2g)$.

Before moving on it is worthwhile to mention the consequence of the conjectured Bogomolov–Miyaoka–Yau inequality for symplectic manifolds to determining the shape of $D_{\mathfrak{M}^{\min}(G)}$. Recall that the BMY conjecture states that $\chi - 3\sigma \geq 0$ for all minimal symplectic manifolds with $K^2 \geq 0$. This gives a lower bound for $\chi - 3\sigma$ on $\mathfrak{M}^{\min}(G)$ whenever that G is not a surface group, and hence implies that in this case $\mathfrak{M}^{\min}(G)$ contains the cone over the interval $\{1\} \times \left[-3, \frac{3}{2}\right]$, improving Proposition 5 for non-surface groups. It is worth noting that all currently known simply-connected irreducible 4-manifolds satisfy $\chi - \frac{3}{2}\sigma \geq 0$.

It is perhaps most natural to describe the domains $D_{\mathfrak{M}}$ as cones on an interval contained in the unit circle and $f_{\mathfrak{M}}$ as functions on these intervals. For example $D_{\mathfrak{M}^\infty}(G)$ corresponds to the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, and $D_{\mathfrak{M}}(G)$ corresponds to the interval $\left[\theta_G, \frac{\pi}{4}\right]$. However, we find it more convenient to describe them in terms of intervals in $\{1\} \times \mathbb{R}$ for two reasons. First, $a\chi + b\sigma$ is not bounded below on $\mathfrak{M}(G)$ for $a \leq 0$. But for $a > 0$ one can divide by a and minimize the 1-parameter family $\chi + b\sigma$ without losing information. Secondly, the function of one variable $b \mapsto f_{\mathfrak{M}}(1, b)$ can easily be shown to be a piecewise linear concave function, and can often be explicitly described.

Thus we restrict to the case $a > 0$ (and hence to $a = 1$ by normalizing) and consider the intersection of the line $\{a = 1\}$ with the domains $D_{\mathfrak{M}^\infty}(G)$, $D_{\mathfrak{M}}(G)$, and $D_{\mathfrak{M}^{\min}}(G)$. Propositions 4 and 5 show that there are natural breaking points at $b = 1$, and $b = \frac{3}{2}$, corresponding to $\chi + \sigma$ and $\chi + \frac{3}{2}\sigma$. The comments after Proposition 5 completely compute the minimum of $\chi + \frac{3}{2}\sigma$ on $\mathfrak{M}^{\min}(G)$. These breaking points really do matter, as the next few calculations of the functions $f_{\mathfrak{M}}$ over the line $a = 1$ show.

Consider first $G = \{e\}$ the trivial group:

$$f_{\mathfrak{M}^\infty(e)}(1, b) = \begin{cases} 2 & \text{if } |b| \leq 1, \\ -\infty & \text{otherwise,} \end{cases}$$

with S^4 the minimizer for all $|b| \leq 1$. By contrast,

$$f_{\mathfrak{M}(e)}(1, b) = \begin{cases} b + 3 & \text{if } |b| \leq 1, \\ -\infty & \text{if } b < -3 \text{ or } b > 1, \\ \text{unknown, but } \leq b + 3 & \text{if } -3 \leq b < -1. \end{cases}$$

with $\mathbb{CP}^2$ the minimizer for all $|b| \leq 1$, and Stipsicz's examples [27] treating the cases $b < -3$.

For $\mathfrak{M}^{\min}(e)$ the domain of $f_{M^{\min}(e)}(1, b)$ includes $1 \leq b \leq \frac{3}{2}$. Considering $\mathbb{CP}^2$, Dolgachev surfaces, and Stipsicz's examples yields the following:

$$f_{\mathfrak{M}^{\min}(e)}(1, b) \begin{cases} \leq b + 3 & \text{if } b < -1, \\ = b + 3 & \text{if } |b| \leq 1, \\ = -8b + 12 & \text{if } 1 \leq b \leq \frac{3}{2}, \\ = -\infty & \text{if } b < -3 \text{ or } b > \frac{3}{2}. \end{cases}$$
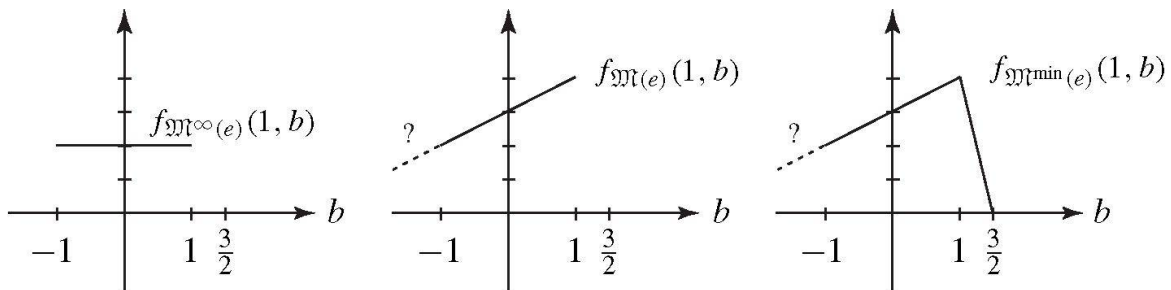
Altogether, the functions yield the following graphs.

Figure 3

Another interesting example is the case of $G = \mathbb{Z}^6$. In [11] it is shown that any smooth oriented 4-manifold M with fundamental group $\mathbb{Z}^6$ has $\chi(M) \geq 6$. Moreover, $b^+(M) \geq 7$. To see this, consider the composite $\pi_1(M) \cong \mathbb{Z}^6 \to \mathbb{Z}^5$, where the second map is the projection to the first five coordinates. The induced map $H^2(\mathbb{Z}^5) \to H^2(M)$ is injective, and Theorem 4 of [11] shows that its image has a 7-dimensional isotropic subspace. Hence $\chi(M) + \sigma(M) \geq 4$.

The symplectic manifold S_3 described below in Section 5 has fundamental group $\mathbb{Z}^6$, $\chi(S_3) = 6$, and $\sigma(S_3) = -2$. Thus $f_{\mathfrak{M}^\infty(\mathbb{Z}^6)}(1, 0) = 6$ and $f_{\mathfrak{M}^\infty(\mathbb{Z}^6)}(1, 1) = 4$. This shows

$$f_{\mathfrak{M}^\infty(\mathbb{Z}^6)}(1, b) = \begin{cases} 6 - 2|b| & \text{if } |b| \leq 1, \\ -\infty & \text{if } |b| > 1. \end{cases} \tag{3}$$

In (3) we used the fact that by reversing orientation shows that $b \mapsto f_{\mathfrak{M}^\infty(G)}(1, b)$ is an even function. This is not true for $\mathfrak{M}(G)$, i.e. for symplectic manifolds, as the example with $G = \{e\}$ above shows.

The domains $D_{\mathfrak{M}^\infty(G)}$ are independent of G, but we do not know the answer to the question:

Question 2. Are the domains $D_{\mathfrak{M}(G)}$ and $D_{\mathfrak{M}^{\min}(G)}$ independent of G?

The examples given above show that the functions $f_{\mathfrak{M}(G)}$ do depend on G in interesting ways.

Motivated by the results of this section we will concentrate on minimizing χ and $\chi + \sigma$ for the rest of the paper.

4. Algebraic upper bounds

We next state and prove two theorems which give algebraically determined bounds in terms of a presentation of G.

Theorem 6. *Let G have a presentation with g generators $x_1, \ldots, x_g$ and r relations $w_1, \ldots, w_r$. Then there exists a symplectic 4-manifold M with $\pi_1 M \cong G$, Euler characteristic $\chi(M) = 12(g + r + 1)$, and signature $\sigma(M) = -8(g + r + 1)$.*

Combining this with the bound (1) one obtains:

Corollary 7. *For a finitely presented group G with g generators and r relations,*

$$2 - 2b_1(G) + b_2(G) \leq \min_{M \in \mathfrak{M}(G)} \chi(M) \leq 12(g + r + 1). \tag{4}$$

and

$$\min_{M \in \mathfrak{M}(G)} \chi(M) + \sigma(M) \leq 4(g + r + 1). \tag{5}$$

$\square$

For specific groups one can (and we will; see below) do better. One general class of groups for which we can improve the construction of Theorem 6 and hence the upper bound in (4) is treated in the following theorem. We will show below that this class includes free groups.

Theorem 8. *Let $H \colon F \to F$ be an orientation-preserving diffeomorphism of an orientable surface F. Assume H fixes a base point z. Let G be the quotient of $\pi_1(F, z)$ by the normal subgroup generated by the words $x^{-1} H_*(x)$, $x \in \pi_1(F, z)$.*

Then there exists a symplectic 4-manifold M with $\pi_1 M \cong G$, Euler characteristic $\chi(M) = 12$, and signature -8.

Proof. We prove Theorems 6 and 8 simultaneously. The arguments we give are derived from Gompf's arguments and follow by combining them with the construction of symplectic forms on $M \times S^1$, where M is a fibered 3-manifold. The flexibility gained by replacing Gompf's choice of $M = F \times S^1$ with a fibered manifold leads to a simplified and ultimately smaller (as measured by the Euler characteristic) construction.

We begin with a discussion of how to put symplectic forms on 4-manifolds of the form $N \times S^1$, where N is a surface bundle over S^1. This construction has its origins in Thurston's article [29].

Let F be an oriented surface. Let $H: F \to F$ be a diffeomorphism with at least one fixed point, and let $p: M \to S^1$ denote the mapping torus of H, fibered over the circle with fiber F and monodromy H.

Let g_0 be a Riemannian metric on F, and let g_t be a path of Riemannian metrics from g_0 to $g_1 = H^*(g_0)$. Then $H: (F, g_0) \to (H, g_1)$ is an isometry.

Notice that if H is an isometry with respect to some metric g_0 then one can take g_t to be the constant path. In this case the volume form of g_0 on F determines a closed 2-form β on M whose restriction to each fiber is a volume form (i.e. a closed, nowhere-zero, top dimensional form).

In general, we find such a 2-form as follows. Let $\alpha_t \in \Omega^2_F$ denote the volume form of the metric g_t and the given orientation. Since H is an orientation-preserving diffeomorphism, the cohomology classes $[\alpha_0]$ and $[\alpha_1]$ in $H^2(F; \mathbb{R}) \cong \mathbb{R}$ are equal. Hence there exists a positive smooth function $f: [0, 1] \to (0, \infty)$ with $f(0) = 1 = f(1)$ so that the cohomology class $[f(t)\alpha_t]$ is independent of t. Denote the closed, nondegenerate 2-form $f(t)\alpha_t$ on F by β_t.

Moser's stability theorem (see [23]) implies that there is a 1-parameter family of diffeomorphisms $\psi_t: F \to F$ so that ψ_0 is the identity and $\psi_t^*(\beta_t) = \beta_0$. The trace $(x, t) \mapsto (\psi_t(x), t)$ induces a diffeomorphism $\Psi: M \to M'$, where M' denotes the mapping torus of $\psi_1 \circ H$.

Let $\pi: F \times [0, 1] \to F$ denote the projection to the first factor. The 2-form β on $F \times [0, 1]$ defined by $\beta = \pi^*(\beta_0)$ is closed. Moreover, since $(\psi_1 \circ H)^*(\beta_0) = H^*(\psi_1^*(\beta_0)) = H^*(\beta_1) = \beta_0$, β descends to a well-defined closed 2-form on M' whose restriction to each fiber is a volume form. Pulling this form back to M via Ψ determines a closed 2-form on M whose restriction to each fiber is a volume form. Denote this 2-form by $\beta \in \Omega^2_M$.

Let dt denote the volume form on the base of the fibration $p: M \to S^1$. Then $p^*(dt)$ is a 1-form on M. Denote by N the 4-manifold $M \times S^1$. To distinguish it from the base of the fibration denote the volume 1-form on the second factor by ds. Let $q_1: M \times S^1 \to M$ and $q_2: M \times S^1 \to S^1$ denote the projections to each factor. Then $q_2^*(ds)$ is a 1-form on N.

The 2-form

$$\omega = q_1^*(\beta) + p^*(dt) \wedge q_2^*(ds) \qquad (6)$$

is a symplectic form on N. Indeed, since β is closed, $d\omega = 0$, and one can check locally that $\omega \wedge \omega$ is nowhere zero.

If z is a fixed point of H, then the circle $z \times_H S^1 \subset M$ determines a torus $T_0 = (z \times_H S^1) \times S^1 \subset M \times S^1 = N$. The restriction of ω to this torus is a volume form; with a slight abuse of notation it is just the form $dt \wedge ds$. Thus T_0 is a symplectic torus in N. Note that the self-intersection number $T_0 \cdot T_0$ in N is zero.

The fundamental group of M is the HNN extension of $\pi_1 F$ with respect to the automorphism induced by H, i.e.

$$\pi_1 M = \langle \pi_1 F, t \mid H_*(x) = txt^{-1} \text{ for each } x \in \pi_1 F \rangle,$$

and $\pi_1 N = \pi_1 M \times \mathbb{Z}$. Denote by s the generator of the second factor. Note that the Euler characteristic and signature of N vanish.

Theorem 8 can now be proved, following Gompf's argument. The group G of Theorem 8 is obtained by taking the quotient of $\pi_1(N)$ by the normal subgroup generated by t and s.

Recall from [6] that if X and Y are symplectic 4-manifolds containing symplectic tori $T_X \subset X$ and $T_Y \subset Y$ with trivial normal bundles then the symplectic sum of X and Y along T_X and T_Y, denoted by $X \#_T Y$, is obtained by removing tubular neighborhoods of T_X and T_Y and identifying the resulting boundaries, which are S^1 bundles over T, by a fiber-preserving, orientation reversing diffeomorphism. The symplectic sum admits a symplectic structure so that any symplectic surface in $X - T_X$ or $Y - T_Y$ remains symplectic in $X \#_T Y$.

Suppose that E is a symplectic 4-manifold which contains a symplectic torus T with self-intersection number zero satisfying $\pi_1(E - T) = 1$. Then Van Kampen's theorem implies that the fundamental group of the symplectic sum $N \#_T E$ along T_0 in N is obtained from the fundamental group of N by killing the image of $\pi_1(T_0)$ in $\pi_1(N)$. Taking E to be the elliptic surface $E(1)$ and T a generic fiber gives the desired symplectic manifold $S = N \#_T E(1)$ with $\pi_1 S = G$, $\chi(S) = 12$, and $\sigma(S) = -8$.

We turn now to the proof of Theorem 6. From the presentation of G with generators $x_1, \ldots, x_g$ and relations $w_1, \ldots, w_r$, construct a new presentation with $2g$ generators $x_1, y_1, \ldots, x_g, y_g$, and $g + r$ relations: the first g relations are $x_1 y_1, \ldots, x_g y_g$ and the last r relations are $w'_1, \ldots, w'_r$. Here w'_i is obtained from w_i by replacing every occurrence of x_j^{-a} for $a > 0$ with y_j^a for all j. The relevant observation for our purposes is that in every relation the generators appear with only positive powers.

Let $T = S^1 \times S^1$ and define $f: S^1 \times S^1 \to S^1$ by $f(e^{ia}, e^{ib}) = e^{i(a+b)}$. Let $X = S^1 \times \{1\}$ and $Y = \{1\} \times S^1$. Let $D \subset T$ be a small 2-disk in the complement of $X \cup Y$. Let $w: T \to T$ be a smooth map that collapses D to a point and is a diffeomorphism on the complement of D. Denote by θ the 1 form on T obtained by pulling back the volume form on S^1, $\theta = w^*(f^*(dt))$. This is a 1-form on T which vanishes on D, and restricts to a volume 1-form on any *positive monotonic* path in $T - D$, that is, any smooth (oriented) path in $T - D$ whose composite with $f \circ w$ wraps monotonically (with non-vanishing derivative) around S^1 in the positive direction.

Let n_i denote the length of the relation w'_i (e.g. the length of $x_5^3 y_1 y_2^2$ is 3). Let

$$n = 1 + \left(\sum_{i=1}^{r} n_i \right).$$

Consider the (isometric) $\mathbb{Z}/(ng)$ action of S^2 generated by the rotation R about the z axis by angle $2\pi/(ng)$. Let D' be a small disc in S^2 centered on the equator (say at $(1,0,0)$) such that its translates by R are all pairwise disjoint. Let F be the orientable surface of genus gn constructed by removing all the translates of D' by powers of R and gluing in one copy of $T - D$ along each boundary circle. There is a corresponding isometry $R: F \to F$ which takes each copy of $T - D$ to the next. The 1-form θ on T defines a smooth 1-form (which we continue to call θ) on F which vanishes outside the union of the $T - D$ and which is invariant under R. Another description of this entire construction is to consider the ng-fold cyclic branched cover of T branched over two points in D and to pull back the 1-form θ to the branched covering.

For convenience denote $F = A \cup B$, where A is the complement of the ng discs $R^k(D')$ in S^2 and B is the disjoint union of the ng punctured tori. The region A is shaded in Figure 4.

Let $H = R^g: F \to F$. Thus H is an isometry of order n. We label the images of the curves X and Y in the various copies of $T - D$ using a double index, $X_{i,j}, Y_{i,j}$, $i = 1, \ldots, g$, $j = 1, \ldots n$ labeled lexicographically. Thus $H(X_{i,j}) = X_{i,j+1}$ and $H(Y_{i,j}) = Y_{i,j+1}$ (with j taken modulo n). In other words, the labeling is lifted from the n-fold branched cover $F \to F/H$.

Join the intersection point of $X_{i,j}$ and $Y_{i,j}$ to the north pole $z = (0,0,1)$ along a great circle to obtain generators $x_{i,j}$ and $y_{i,j}$ of $\pi_1(F, z)$. Thus the induced action on $\pi_1 F$ is given by $H_*(x_{i,j}) = x_{i,j+1}$ and $H_*(y_{i,j}) = y_{i,j+1}$.

To the ordered set of relations $w_1', w_2', \ldots, w_r'$ we assign an ordered set of words $\tilde{w}_1, \ldots, \tilde{w}_r$ in the $x_{i,j}$ and $y_{i,j}$ as follows. Starting with the first letter which appears in w_1' replace the corresponding x_i or y_i by $x_{i,1}$ or $y_{i,1}$. For the second letter which appears in w_2' add the second index 2 to its subscript, and continue until all the letters in w_1' are replaced by doubly indexed letters in such a way that as the word is read from left to right, the second indices increase. Then proceed to the second relation w_2', and so forth. Thus when the words $\tilde{w}_1, \ldots, \tilde{w}_r$ are read from left to right, the second index in the subscripts will read "$1, 2, \ldots, n - 1$".

For example, this process converts the set of relations

$$(w_1', w_2') = (y_2 x_1^3 x_5, y_4 y_3^2)$$

to

$$(\tilde{w}_1, \tilde{w}_2) = (y_{2,1} x_{1,2}^3 x_{5,3}, y_{4,4} y_{3,5}^2).$$

From the $\tilde{w}_i$ one can easily construct *pairwise disjoint immersed* curves $\gamma_i: S^1 \to F$ for $i = 1, \ldots, r$ with the properties:

(1) γ_i (connected to the north pole along a great circle) represents the word $\tilde{w}_i$ in $\pi_1(F, z)$.
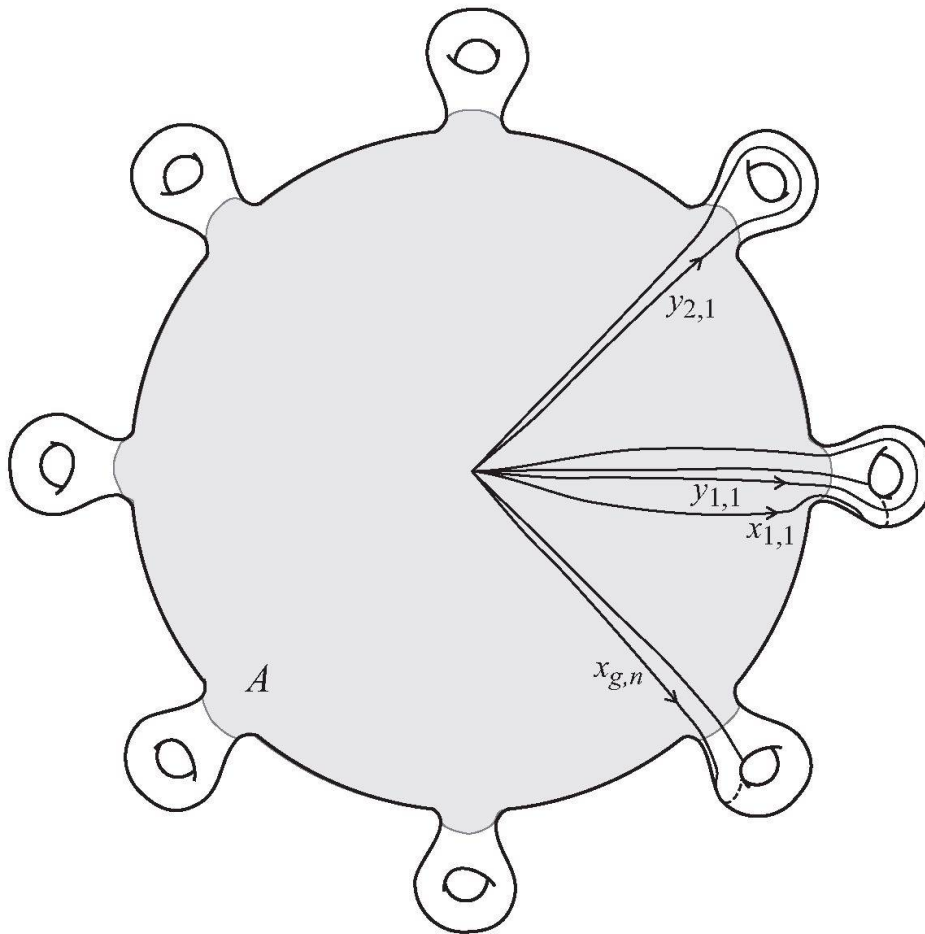
Figure 4

(2) The double points (if any) of γ_i are finite, transverse and contained entirely in B.

(3) γ_i restricts to a positive monotonic path in each component (i.e. punctured torus) of B. (This is where we use the fact that the relations involve only positive powers of the x_i and y_i.)

(4) The curves γ_i intersect each component (i.e. circle) of $A \cap B$ transversely.

The pulled back 1-form $\gamma_i^*(\theta)$ is a positive multiple of dt on that part of S^1 mapped into the interior of B by γ_i and is zero on $\gamma_i \cap A$. One can find a function f_i on $\gamma_i^{-1}(A)$ so that f_i vanishes on the endpoint of each arc in $\gamma_i^{-1}(A)$ and so that $\gamma_i^*(\theta) + df_i$ is a volume form on S^1. Since the intersection of the union of the γ_i with A is a collection of pairwise disjoint embedded arcs, one can extend each f_i to a function on F which vanishes outside a neighborhood of $\gamma_i \cap A$ and vanishes on B. Adding

the values of the f_i yields a function $f: F \to \mathbb{R}$ so that $\gamma_i^*(\theta + df)$ is a volume (i.e. nowhere vanishing) 1-form on S^1 for each i.

We also need g extra curves, corresponding to the relations $x_i y_i$, $i = 1, \ldots, g$. Notice that n was the sum of the lengths of the relations, plus one. We use this extra bit of surface to construct immersions (in fact these can be taken to be embeddings) $\gamma_{r+k}: S^1 \to F$, $k = 1, \ldots, g$ corresponding to the words $x_{1,n} y_{1,n}, \ldots, x_{g,n} y_{g,n}$. These curves can each be taken to lie entirely in one punctured torus component of B and be positive and monotonic in this component. (Alternatively, we could have made n larger and treated these relations exactly as we did with the first type of relation. We choose this approach since our intention is to find as small a universal construction as possible.) The 1-form df vanishes on these last g punctured tori by construction, and so $\gamma_i^*(\theta + df)$ is a volume form for $i = r + 1, \ldots, r + g$ as well.

Since the form θ is invariant under H, the pull back $\pi_1^*(\theta)$ via the projection $\pi_1: F \times [0, 1] \to F$ is a closed 1-form on $F \times [0, 1]$ which determines uniquely a well-defined 1-form Θ on the mapping torus $M = F \times_H S^1$ of H with the property that the restriction of Θ to $F \times \{0\} \subset M$ equals θ. The function $f: F = F \times \{0\} \to \mathbb{R}$ extends to a function (still called f) on M (say by using a cut-off function in the interval coordinate). Thus we end up with a closed 1-form $\Theta + df$ on M whose restriction to the fiber $F \times \{0\}$ pulls back to a volume form for each $\gamma_i: S^1 \to F$.

Let $N = M \times S^1$. For small enough ε, the form

$$\omega_\varepsilon = q_1^*(\beta) + p^*(dt) \wedge q_2^*(ds) + \varepsilon q_1^*(\Theta + df) \wedge q_2^*(ds)$$

is a symplectic form on N. For each $i = 1, \ldots, r+g$ the immersed torus $T_i = \gamma_i \times S^1$ is Lagrangian with respect to $q_1^*(\beta) + p^*(dt) \wedge q_2^*(ds)$. Since $q_1^*(\Theta + df) \wedge q_2^*(ds)$ is a volume form on $\gamma_i \times S^1$, the T_i are symplectic with respect to ω_ε for small positive ε. The T_i can be regularly homotoped to embeddings by a small regular homotopy by separating the double points of γ_i using the parameter transverse to the fibers in the fibration of M. Pushing the curve γ_i into a far away fiber can be used to construct a homotopy of T_i off itself. Thus the T_i have self-intersection zero.

Finally, we saw before that the "vertical torus" $T_0 = T = z \times_H S^1$ is symplectic with respect to $\omega = \omega_0$; hence it remains symplectic with respect to ω_ε for small enough ε.

The fundamental group of N is generated by the $x_{i,j}$, $y_{i,j}$, t, and s subject to the relations:

$$\prod_{i,j} [x_{i,j}, y_{i,j}] = 1, \quad t x_{i,j} t^{-1} = x_{i,j+1}, \quad t y_{i,j} t^{-1} = y_{i,j+1}, \quad s \text{ is central.}$$

It follows that the quotient of $\pi_1 N$ obtained by killing the generators s, t, the words $\tilde{w}_i$ and $x_{i,n} y_{i,n}$ has the presentation with generators x_i, $i = 1, \ldots, g$ and relations w_i.

Thus to complete the argument we form the symplectic sum of N with $g + r + 1$ copies of the elliptic surface $E(1)$ along the symplectic tori $T_0, T_1, \ldots, T_{r+g}$. Sum-

ming along T_0 kills t and s. Summing along T_i, $i = 1, \ldots, r$ kills $\tilde{w}_i$, and summing along $T_{r+1}, \ldots, T_{r+g}$ sets $x_{i,j}$ equal to $y_{i,j}$. Note that this kills the commutator $[x_{i,j}, y_{i,j}]$ and hence the surface relation disappears. A simple calculation using the Mayer–Vietoris sequence and Novikov additivity shows that each sum increases χ by 12 and decreases σ by 8, completing the proof.   □

Notice that the manifold M constructed in the proof of Theorem 6 is fibered over S^1 with finite order monodromy and with two fixed points. It follows that M is Seifert-fibered over a surface S of genus g with two singular fibers. If $s \colon M \to S$ denotes the Seifert fibration, then the composite of the projection $M \times S^1 \to M$ and $s \colon M \to S$ is a singular fibration with torus fibers. The torus T_0 is one of the singular fibers. Nearby smooth fibers form an n fold cover of T_0. The tori T_i are products of curves γ_i in a section of the Seifert fibration with the last S^1 factor.

The proof of Theorem 6 also proves the following, which is useful for certain classes of groups.

Corollary 9. *Let G be the quotient of a surface group $\langle x_i, y_i \mid \prod_i [x_i, y_i] \rangle$ by a normal subgroup generated by n words $w_1, \ldots, w_n$ in which the x_i and y_i appear with only positive exponents. Then there is a closed symplectic 4-manifold with fundamental group G, Euler characteristic $12(n + 1)$ and signature $-8(n + 1)$.*   □

A very interesting question is whether the number 12 which occurs in Theorems 6 and 8, and Corollaries 7 and 9 can be improved. Suppose that E is a symplectic manifold which contains a symplectic torus $T \subset E$ such that $T \cdot T = 0$, and so that $\pi_1(E - T) = 1$. Then if $k = \chi(E)$, the number 12 in these theorems can be replaced by k.

We can require even less: suppose that K is a symplectic manifold which contains a symplectic torus $T \subset K$ such that $T \cdot T = 0$ and so that $\pi_1(K - T) = \mathbb{Z}$. Let $p \colon T \to K - T$ denote a push off of T into the boundary of its tubular neighborhood. Suppose that the induced homomorphism $p_* \colon \pi_1(T) \to \pi_1(K - T)$ is surjective. Notice that p_* contains a primitive vector in its kernel, and so symplectically summing with K can be used just as $E(1)$ was used in the proof. If $\chi(K) = \ell$, then the $12(g + r + 1)$ which occurs in Theorem 6 can be replaced by $\ell(g + r + 2)$ or $\ell(g + r) + k$, with k as in the previous paragraph. This is because the first symplectic sum used in the proof of Theorem 6 (along T_0) is used to kill two generators, t and s, whereas the subsequent sums only need to kill one generator at a time.

We summarize these observations in the following corollary for completeness.

Corollary 10. *Let E be a closed symplectic 4-manifold which contains a symplectically embedded torus T with self-intersection zero such that $\pi_1(E - T)$ is trivial and with $\chi(E) = k$. Let K be a closed symplectic 4-manifold which contains a symplectically embedded torus T with self-intersection zero such that $\pi_1(K - T) \cong \mathbb{Z}$,*

$p_* : \pi_1(T) \to \pi_1(K - T)$ *surjective, and* $\chi(K) = \ell$. *Then if G admits a presentation with* g *generators and* r *relations,*

$$\min_{M \in \mathfrak{M}(G)} \chi(M) \leq k + \ell(g + r). \tag{7}$$

$\square$

Unfortunately, we do not know of any "small" examples of E or K as above. The smallest example of such an E we know is $E(1)$. The adjunction inequality ([16]) can be used to show that any such E must have $\chi(E) \geq 6$. Since our constructions are based on taking symplectic sums with $E(1)$, the smallest example we know of a K as in Corollary 10 has $\chi(K) = 12$ (see Lemma 18 below).

5. Bounds for specific classes of groups

In this section we derive better bounds for free groups, cyclic groups, and free abelian groups than those given in Corollary 4. In particular, we determine the lower bound for certain free abelian groups and provide an example of a minimizer.

5.1. Free groups.

Theorem 11. *For any finitely generated free group* G *there exists a symplectic* 4-*manifold* M *with fundamental group* G *and* $\chi(M) = 12$, $\sigma(M) = -8$.

Proof. Let F be a surface of genus g. Let X_i, Y_i, $i = 1, \ldots, g$ be a collection of embedded curves forming a standard symplectic basis for $H_1(F)$. Let x_i, $y_i \in \pi_1(F)$ be the corresponding loops obtained by connecting the X_i, Y_i to a base point. Take $H : F \to F$ to be the composite of Dehn twists along the curves $Y_1, Y_2, \ldots, Y_g$. Then $H_*(x_i) = x_i y_i$ and $H_*(y_i) = y_i$. It follows that the quotient of $\pi_1(F)$ by the normal subgroup generated by $x^{-1} H_*(x)$, $x \in \pi_1 F$ is free with generators $x_1, \ldots, x_g$. Applying Theorem 8 finishes the argument. $\square$

Corollary 12. *Let* G *denote the free group on* n *generators. Let* $e = 0$ *if* n *is even and* $e = 1$ *if* n *is odd. Then*

$$3 - 2n + e \leq \min_{M \in \mathfrak{M}(G)} \chi(M) \leq 12,$$

and

$$4 - 2n + 2e \leq \min_{M \in \mathfrak{M}(G)} \chi(M) + \sigma(M) \leq 4.$$

Proof. Theorem 13 establishes the upper bounds. Let M be symplectic with $\pi_1(M) \cong G$. Notice that $\chi(M) = 2 - 2n + b^+(M) + b^-(M)$ and $\chi(M) + \sigma(M) = 2 - 2n + 2b^+(M)$. Since M is symplectic, $b^+(M) \geq 1$. Moreover, since $1 - b_1(M) + b^+(M)$ is even, $b^+(M)$ is even if n is odd, so that for n odd $b^+(M) \geq 2$. $\qquad\square$

Notice that for $G \cong \mathbb{Z}$ the upper and lower bounds in the second formula of Corollary 12 coincide. Thus our construction gives a symplectic 4-manifold with fundamental group $\mathbb{Z}$ which minimizes $\chi + \sigma$.

Kotschick [15] improves the lower bound for min χ in Corollary 12 from $3 - 2n + e$ to $\frac{6}{5}(1 - n)$ using the fact that $2\chi + 3\sigma \geq 0$ for minimal symplectic 4-manifolds with free fundamental groups.

5.2. Cyclic groups.
We begin with an estimate for cyclic groups which uses Theorem 8. The argument we give is identical to the argument given by Gompf in Proposition 6.4 of [6].

Theorem 13 (Gompf). *There exists a symplectic 4-manifold M with fundamental group $G \cong \mathbb{Z}/n$ satisfying $\chi(M) = 12$ and $\sigma(M) = -8$.*

Proof. Let F be a torus. Take $H \colon F \to F$ to be diffeomorphism which induces the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 2-n \end{pmatrix}$$

on $\mathbb{Z}^2 = H_1(F) = \pi_1(F)$. The quotient of $\pi_1(F)$ by the normal subgroup generated by $x^{-1}H_*(x)$, $x \in \pi_1 F$ is isomorphic to $\mathbb{Z}/n$, since elementary row and column operations transforms $H_* - I$ to the diagonal matrix with entries n and 1. Applying Theorem 8 finishes the argument. $\qquad\square$

Corollary 14. *Let $G = \mathbb{Z}/n$ for $n \neq 0$. Then*

$$3 \leq \min_{M \in \mathfrak{M}(G)} \chi(M) \leq 12,$$

and

$$\min_{M \in \mathfrak{M}(G)} \chi(M) + \sigma(M) = 4.$$

Proof. If M is symplectic with $\pi_1(M) \cong \mathbb{Z}/n$, then $\chi(M) = 2 + b_2(M) \geq 2 + b^+(M) \geq 3$. Moreover, $\chi(M) + \sigma(M) = 2 + 2b^+(M) \geq 4$. The upper bounds come from Theorem 13. $\qquad\square$

Notice that if M denotes the algebraic surface obtained from $E(1)$ by performing two logarithmic transformations of multiplicity p, q with $n = \gcd(p, q)$, then

$\pi_1(M) = \mathbb{Z}/n$, $\chi(M) = 12$, and $\sigma(M) = -8$. This shows that Theorem 13 can be improved: one can replace "symplectic 4-manifold" by "Kähler surface."

The examples of Theorem 12 do not always minimize the Euler characteristic. For example, there are smooth complex projective surfaces with fundamental group $\mathbb{Z}/5$ (Catanese) and $\mathbb{Z}/8$ (Reid) with $\chi = 10$. There are smooth complex projective surfaces with fundamental group $\mathbb{Z}/2$ (Barlow and Reid) and $\mathbb{Z}/4$ (Godeaux) with $\chi = 11$. These examples have $\chi + \sigma = 4$ [1].

5.3. Free abelian groups.
We turn to some calculations and estimates of the minimal values of χ, $\chi + \sigma$ on $\mathfrak{M}(G)$ for G free abelian.

Recall first that smooth 4-manifolds were constructed in [11] which minimize $\chi(M)$ over the class of *smooth* manifolds M with $\pi_1(M) = \mathbb{Z}^n$. It was shown that the minimal Euler characteristic for $n \neq 3, 5$ is

$$2 - 2n + C(n, 2) + \varepsilon_n,$$

where $C(n, 2)$ denotes the binomial coefficient $n(n-1)/2$, and ε_n is 1 if $C(n, 2)$ is odd and zero otherwise. For $n = 3$ (resp. $n = 5$) the minimal Euler characteristic is 2 (resp. 6). We will show below that for n even virtually the same result holds if we minimize over the class of symplectic 4-manifolds. For n odd the situation is less clear.

We begin by setting up some notation and making some easy observations. Let $G = \mathbb{Z}^n$ and let M be a smooth, closed 4-manifold with $\pi_1(M) \cong G$. Choose a map $f: M \to T^n$ inducing an isomorphism on fundamental groups. Since the cohomology ring $H^*(T^n)$ is an exterior algebra on $H^1(T^n)$, the induced map $f^*: H^2(T^n) \to H^2(M)$ is (split) injective. In particular $2 - 2n + C(n, 2) \leq \chi(M)$. Moreover, $\chi(M) + \sigma(M) = 2 - 2n + 2b^+(M) \geq 2 - 2n$.

Note that $\mathbb{Z}^n$ contains subgroups isomorphic to $\mathbb{Z}^n$ of arbitrarily large finite index. Since χ and σ are multiplicative with respect to finite covers,

$$0 \leq \chi(M) + \sigma(M). \tag{8}$$

We turn now to the search for symplectic examples which minimize χ and $\chi + \sigma$.

Proposition 15. *Any closed symplectic 4-manifold M with $\pi_1(M) \cong \mathbb{Z}^n$ satisfies*

$$\chi(M) \geq \begin{cases} 2 - 2n + C(n, 2) & \text{if } n \equiv 1 \text{ or } 4 \bmod 8, \\ 3 - 2n + C(n, 2) & \text{otherwise} \end{cases}$$

and

$$\chi(M) + \sigma(M) \equiv 0 \bmod 4.$$

Proof. The cases $n = 0, 1, 2$ are easy, so we assume that $n \geq 3$. Suppose that M is a closed symplectic 4-manifold with $\pi_1(M) \cong \mathbb{Z}^n$. Then $\chi(M) = 2 - 2n + b_2(M)$. Since M is symplectic, $1 - b_1(M) + b^+(M) = 1 - n + b^+(M)$ is even. Hence $2 - 2n + 2b^+(M) = \chi(M) + \sigma(M) \equiv 0 \bmod 4$.

The bound (1) (or see the paragraph preceding Equation 8) implies that $b_2(M) \geq C(n, 2)$. The theorem will follow if we can show that this bound can be improved to $b_2(M) \geq C(n, 2) + 1$ when n is not congruent to 1 or 4 mod 8.

Assume that $b_2(M) = C(n, 2)$.

As remarked in [11], if $b_2(M) = C(n, 2)$, the injection $f^*\colon H^2(T^n) \to H^2(M)$ is an isomorphism. Since $H^*(T^n)$ is an exterior algebra (over $\mathbb{Z}$) on $H^1(T^n)$, $H^2(T^n)$ has a basis for which each basis vector has cup square zero. This forces the intersection form of M to be even and hence have even rank. Thus $C(n, 2)$ is even.

This proves that $b_2(M) \geq C(n, 2) + 1$ whenever $C(n, 2)$ is odd, i.e. if $n = 4k + 2$ or $n = 4k + 3$. (Notice that we did yet not use the fact that M was symplectic.)

Continue with the assumption that $b_2(M) = C(n, 2)$, so that $C(n, 2)$ is even. Since we are assuming that $b_2(M) = C(n, 2)$, the intersection form of M is even, and hence its signature is divisible by 8. Thus $\chi(M) \equiv 0 \bmod 4$, i.e.

$$2 - 2n + C(n, 2) \equiv 0 \bmod 4 \qquad (9)$$

A simple calculation establishes that if $n = 4k$, then Equation (9) forces k to be odd. Similarly, if $n = 4k + 1$, then k must be even. Thus we have shown that with the possible exception of $n = 8k + 1$ and $n = 8k$, a symplectic 4-manifold M with $\pi_1(M) \cong \mathbb{Z}^n$ must have $b_2(M) \geq C(n, 2) + 1$, and so $\chi(M) \geq 2 - 2n + C(n, 2) + 1$. $\square$

In [11] it was shown that there exist smooth closed 4-manifolds X_n with $\pi_1(X_n) \cong \mathbb{Z}^n$ and $\chi(X_n) = 2 - 2n + C(n, 2)$ for any $n > 5$ with $C(n, 2)$ even. It follows from Theorem 15 that X_n cannot admit a symplectic structure when $n = 8k$ or $n = 8k + 5$. For these examples, $b_1(X_n)$ is even, $\sigma(X_n) = 0$ and $2\chi(X_n) + 3\sigma(X_n) \geq 0$.

As explained in [11], the cases $\mathbb{Z}^3$ and $\mathbb{Z}^5$ are exceptional. The intersection form of any smooth manifold M with fundamental group $\mathbb{Z}^3$ has a 3-dimensional metabolizer, hence $b^+(M) \geq 3$ and $b^-(M) \geq 3$. If M is symplectic then $b^+(M)$ is even, hence at least 4. Thus $\chi(M) \geq 3$ and $\chi(M) + \sigma(M) \geq 4$. Similarly, the intersection form of any smooth manifold M with fundamental group $\mathbb{Z}^5$ has a 7-dimensional metabolizer. If M is symplectic this implies $\chi(M) \geq 7$ and $\chi(M) + \sigma(M) \geq 8$.

We next look at upper bounds. As a first estimate, since $\mathbb{Z}^n$ has a presentation with n generators and $C(n, 2)$ relations, Theorem 6 and Proposition 15 give the estimates

$$\tfrac{1}{2}(n^2 - 5n + 4) \leq \min_{M \in \mathfrak{M}(\mathbb{Z}^n)} \chi(M) \leq 6(n^2 + n + 2).$$

Thus we see that $\min_{M \in \mathfrak{M}(\mathbb{Z}^n)} \chi(M)$ grows quadratically in n, with leading coefficient

between $\frac{1}{2}$ and 6. It follows from the calculations below that the leading coefficient is $\frac{1}{2}$; in fact $\min_{M \in \mathfrak{M}(\mathbb{Z}^n)} \{\chi(M) - \frac{1}{2}n^2\}$ grows at most linearly in n.

For each integer $g \geq 0$ let F_g denote the surface of genus g. Let $S_g = \mathrm{Sym}^2(F_g)$.

Proposition 16. *The space S_g is a compact Kähler manifold, and in particular is symplectic. Moreover, $\pi_1(S_g) = \mathbb{Z}^{2g}$, $H^2(S_g) = \mathbb{Z}^{C(2g,2)+1}$ so that $\chi(S_g) = 3 - 2(2g) + C(2g, 2)$, and $\sigma(S_g) = 1 - g$.*

Sketch of proof. The fact that S_g admits a complex structure comes from the fact that the $\mathbb{Z}/2$ action on $F_g \times F_g$ defines a branched cover of S_g. The fundamental group is computed using Van Kampen's theorem, splitting S_g along the circle bundle over the branch set; note that the two sets of generators of $\pi_1(F_g \times F_g) \cong \pi_1(F_g) \times \pi_1(F_g)$ commute, and are identified in $\pi_1(S_g)$. Since $b_1(S_g)$ is even, S_g is Kähler [7].

The Riemann–Hurwitz formula computes $\chi(S_g)$ and with the universal coefficient theorem this implies the computation for $H^2(S_g)$. Computing the signature is a bit more involved; the most straightforward way to do this is to use the transfer (with $\mathbb{R}$ coefficients) to observe that the induced map $H^2(S_g) \to H^2(F_g \times F_g)$ is injective with image the $\mathbb{Z}/2$-invariant classes, and to compute the intersection form directly by restricting the intersection form of $F_g \times F_g$.

We refer to [22] for details.   □

The following corollary computes the minimal Euler characteristic for most $\mathbb{Z}^{2g}$.

Corollary 17. *Let $G = \mathbb{Z}^{2g}$.*

(1) *If $g \equiv 0, 1$ or $3 \mod 4$, then*

$$\min_{M \in \mathfrak{M}(G)} \chi(M) = 3 - 4g + C(2g, 2),$$

with minimizer S_g.

(2) *If $g \equiv 2 \mod 4$, then*

$$0 \leq \min_{M \in \mathfrak{M}(G)} \chi(M) - \big(2 - 4g + C(2g, 2)\big) \leq 1.$$

(3) $0 \leq \min_{M \in \mathfrak{M}(G)} \chi(M) + \sigma(M) \leq 4 - 5g + C(2g, 2).$

Proof. The examples S_g of Proposition 16 provide the upper bounds. Proposition 15 shows that when $g \equiv 0, 1$ or $3 \mod 4$, the S_g give the smallest possible χ. When $g \equiv 2 \mod 4$, the lower bound of Proposition 15 differs by one from $\chi(S_g)$. The third assertion comes from Equation 8 and Proposition 16.   □

Corollary 17 does not answer the question of whether S_g minimizes χ on $\mathfrak{M}(\mathbb{Z}^{2g})$ when $g = 4m + 2$. In fact it does not for $m = 0$: $S_2 = T^4 \# \overline{\mathbb{CP}}^2$. But the 4-torus T^4 is symplectic and

$$0 = \chi(T^4) < \chi(S_2) = 1.$$

We do not know whether S_{4m+2} minimizes $\chi : \mathfrak{M}(\mathbb{Z}^{8m+4}) \to \mathbb{Z}$ for $m > 0$.

Note that S_2 does minimize $\chi + \sigma$. In fact, S_0, S_1, S_2, and S_3 minimize $\chi + \sigma$ for symplectic manifolds and $G = 0, \mathbb{Z}^2, \mathbb{Z}^4$ and $\mathbb{Z}^6$. The first unknown case is S_4, with $\chi(S_4) + \sigma(S_4) = 12$. Hence either S_4 minimizes $\chi + \sigma$ among symplectic 4-manifolds with fundamental group $\mathbb{Z}^8$ or else (since $\chi + \sigma \equiv 0 \bmod 4$) there is a symplectic 4-manifold X with $\pi_1(X) = \mathbb{Z}^8$ and $b^+(X) = 7, 9$ or 11.

Free abelian groups of odd rank pose a greater challenge. For $G = \mathbb{Z}$, we know that any symplectic 4-manifold M with $\pi_1(M) \cong \mathbb{Z}$ has $b^+(M)$ even and greater than zero, thus $\chi(M) = b_2(M) \geq 2$. On the other hand, Theorem 13 constructs a symplectic 4-manifold with $\pi_1(M) \cong \mathbb{Z}$ with $\chi(M) = 12$ and $\sigma(M) = -8$. At the moment this is the smallest example known to the authors of a symplectic 4-manifold with fundamental group $\mathbb{Z}$ (see Theorem 13). Thus

$$2 \leq \min_{M \in \mathfrak{M}(\mathbb{Z})} \chi(M) \leq 12. \tag{10}$$

This example does minimize $\chi + \sigma$. Indeed, since $b^+(M)$ is even and greater than zero for a symplectic 4-manifold with fundamental group $\mathbb{Z}$, it follows that $\chi(M) + \sigma(M) = 2b^+(M) \geq 4$. The example of Theorem 13 with $\pi_1(M) \cong \mathbb{Z}$ has $\chi(M) + \sigma(M) = 4$, so

$$\min_{M \in \mathfrak{M}(\mathbb{Z})} \chi(M) + \sigma(M) = 4. \tag{11}$$

We turn to the case $G \cong \mathbb{Z}^3$.

Consider the four-torus $X = T^2 \times T^2$ with the product symplectic structure. Its fundamental group is $\mathbb{Z}^4$ generated by the coordinate circles; call these generators a, b, c, d. The Euler characteristic of X is 0 and $\sigma(X) = 0$. The symplectic torus $T_0 = p \times T^2$ has fundamental group generated by c and d and self-intersection 0. We can use the manifold in the next lemma to kill one of the generators c or d.

Lemma 18. *There exists a symplectic 4-manifold K with $\pi_1(K) \cong \mathbb{Z}$ which contains a symplectically embedded torus T with self-intersection zero such that*

(1) $\chi(K) = 12$ and $\sigma(K) = -8$, and

(2) $\pi_1(K - T) \cong \mathbb{Z}$ and the map induced by inclusion $\pi_1(K - T) \to \pi_1(K)$ is an isomorphism.

Proof. Let K be the symplectic 4-manifold with $\pi_1(K) \cong \mathbb{Z}$ constructed in Theorem 13. The construction of K was the following. First a fibered 3-manifold M with fiber a torus F is constructed as the mapping torus of the Dehn twist $H : F \to F$ on the torus along the second curve y of a symplectic basis $\{x, y\}$ of $\pi_1(F)$. Thus $\pi_1(M) = \langle x, y, t \mid [x, y], txt^{-1} = xy, tyt^{-1} = y \rangle$ and letting $N = M \times S^1$, $\pi_1(N) = \langle x, y, t, s \mid [x, y], txt^{-1} = xy, tyt^{-1} = y, s \text{ central} \rangle$. Then N admits a symplectic form ω (see Equation (6)) for which the torus $T_0 = t \times s$ is symplectic, and taking the symplectic sum of N with $E(1)$ along T_0 yields K. Since $\pi_1(N - T_0) \to \pi_1(N)$ is surjective and $\pi_1(E(1) - T_0') = 1$, where T_0' is the elliptic fiber in $E(1)$ along which the symplectic sum is taken, it follows that $\pi_1(K)$ is infinite cyclic, generated by x.

Let T denote the embedded torus in N given by $x \times s$. More precisely, choose an embedded curve γ freely homotopic to x in the fiber F which avoids the base point. Then $T = \gamma \times S^1 \subset M \times S^1$ is a torus and the morphism induced by inclusion takes the two generators of $\pi_1(T)$ to x and s. From Equation (6) one sees that T is Lagrangian in N. Notice also that T is disjoint from $T_0 = t \times s$ since γ avoids the base point of F. Also notice that T has self intersection zero since γ can pushed off itself.

Since x is non-zero in $H_1(M)$, T is non-zero in $H_2(N)$ by the Künneth theorem. It follows by a standard argument (see e.g. Lemma 1.6 of [6]) That ω can be perturbed by an arbitrarily small amount so that T is symplectic with respect to the resulting symplectic form ω'. If the perturbation is taken very small, T_0 remains symplectic. Gompf shows furthermore that a symplectic structure on the symplectic sum $K = N \#_{T_0} E(1)$ can be chosen so that T remains symplectic in K.

Thus $T \subset K$ is a symplectic torus with self-intersection zero for which the induced map on fundamental groups is the map $\mathbb{Z}x \oplus \mathbb{Z}s \to \mathbb{Z}x$, i.e. $x \mapsto x, s \mapsto 1$. To compute $\pi_1(K - T)$, first notice that $N - T = (M - \gamma) \times S^1$. Since γ is a curve in the fiber of the fibration $M \to S^1$ (representing x), it follows that $M - \gamma$ is obtained from $F \times [0, 1]$ by gluing the ends along an annulus, namely the annulus in the torus F complementary to γ. Thus $\pi_1(M - \gamma) = \langle x, y, t \mid [x, y], txt^{-1} = xy \rangle$. It follows that $\pi_1(N - T) = \langle x, y, t, s \mid [x, y], txt^{-1} = xy, s \text{ central} \rangle$. Since $K - T$ is the symplectic sum of $N - T$ with $E(1)$ along T_0,

$$\pi_1(K - T) = \langle x, y, t, s \mid [x, y], \ txt^{-1} = xy, \ s \text{ central}, \ s = 1, \ t = 1 \rangle = \mathbb{Z}x.$$

$\square$

Take the symplectic sum, L, of K and T^4 along T in K and T_0 in T^4,

$$L = T^4 \#_T K,$$

identifying x with c and s with d. This, in effect, kills d without introducing any new relations, giving a symplectic manifold with fundamental group $\mathbb{Z}^3$, with $\chi(L) = 12$

and $\sigma(L) = -8$. Together with the remarks after Proposition 15 this implies the following.

Proposition 19. *There exists a symplectic 4-manifold L with fundamental group $G = \mathbb{Z}^3$ satisfying $\chi(L) = 12$ and $\sigma(L) = -8$. Hence*

$$3 \le \min_{M \in \mathcal{M}(\mathbb{Z}^3)} \chi(M) \le 12.$$

Moreover,

$$\min_{M \in \mathcal{M}(\mathbb{Z}^3)} \chi(M) + \sigma(M) = 4. \qquad \square$$

Finally, we treat the case of odd rank free abelian groups.

Theorem 20. *There exists a symplectic 4-manifold M with $\pi_1(M) \cong \mathbb{Z}^{2n-1}$ such that $\chi(M) = 15 - 5n + 2n^2$ and $\sigma(M) = -7 - n$.*

Theorem 20 gives the bound

$$\min_{M \in \mathfrak{M}(\mathbb{Z}^{2n-1})} \chi(M) - \left(2 - 2(2n-1) + C(2n-1,2)\right) \le 2n + 10.$$

In other words, the difference between the lower bound of Equation (1) and the examples constructed here grows linearly with the rank. This is in contrast with the examples of even rank free abelian groups: that difference is always a constant. On the other hand, it is an improvement over the general construction of Theorem 6, whose difference grows quadratically in n.

The proof of Theorem 20 depends on finding a suitable symplectic form on the (Kähler) manifold $S_g = \mathrm{Sym}^2(F_g)$ for which we can identify certain tori as Lagrangian. The main technical result needed is the following proposition, whose proof was suggested to us by R. Gompf.

Proposition 21. *Let F denote the closed surface of genus g, and let $\pi : F \times F \to S_g$ denote the regular 2-fold branched cover corresponding to the $\mathbb{Z}/2$ action $(x, y) \mapsto (y, x)$ on $F \times F$ with fixed submanifold $B = \{(x, x) \mid x \in F\}$. Let ω_F be a fixed symplectic form on the surface F and let $\omega = \omega_F \oplus \omega_F \in \Omega^2(F \times F)$ be the $\mathbb{Z}/2$-equivariant symplectic form on the product.*

Then there exists a symplectic form $\omega' \in \Omega^2(S_g)$ so that the pullback $\pi^*(\omega')$ agrees with ω outside a small tubular neighborhood of B.

Proof. We first show that in any neighborhood of B in $F \times F$ one can find a tubular neighborhood N of B which admits a semi-free Hamiltonian S^1 action with fixed set B, such that the $\mathbb{Z}/2$ action $(x, y) \mapsto (y, x)$ embeds in the S^1 action as multiplication by -1. The Hamiltonian function $\mu : N \to [0, \varepsilon)$ satisfies $\mu^{-1}(0) = B$. This

is a standard fact in symplectic topology; we include a proof for the benefit of the reader.

Fix a Riemannian metric on $F \times F$ and let $P \to B$ be the principal $SO(2) = U(1)$ bundle associated to the normal bundle of B, i.e. P is the bundle with $c_1(P) = 2 - 2g$. Then P admits a free $\mathbb{Z}/2$ action commuting with the $U(1)$ action, namely multiplication by $-1 \in U(1)$. Let $E = P \times_{U(1)} D^2 \to B$ be the associated disc bundle. Note that E is diffeomorphic to the normal disc bundle ν of $B \subset F \times F$. Moreover, one can choose the diffeomorphism $E \cong \nu$ equivariant with respect to the $\mathbb{Z}/2$ action on E and the linearization of the $\mathbb{Z}/2$ action $(x, y) \mapsto (y, x)$ near B in $F \times F$.

The symplectic form ω on $F \times F$ restricts to a symplectic form ω_B on B. This form extends to an S^1-equivariant symplectic form on E with corresponding Hamiltonian function $\mu \colon E \to [0, 1]$, so that $\mu^{-1}(0) = B$ (see [23, page 155]).

Since E and ν are equivariantly diffeomorphic symplectic bundles and restrict to the same symplectic form ω_B on B, Weinstein's symplectic tubular neighborhood theorem (see [30] and [23, page 98]) implies that there is a $\mathbb{Z}/2$-equivariant symplectomorphism from a neighborhood of the zero section in E and a neighborhood of B in $F \times F$. Since any neighborhood of the zero section in E contains a smaller neighborhood of the form $E_\varepsilon = \mu^{-1}([0, \varepsilon))$, pulling back μ and the $U(1)$ action via the symplectomorphism restricted to E_ε gives the desired neighborhood N, Hamiltonian μ, and corresponding Hamiltonian S^1 action.

Denote the quotient of $N - B$ by the $\mathbb{Z}/2$ action by U. Thus U is endowed with the quotient symplectic structure (since $\mathbb{Z}/2$ acts freely and symplectically on $N - B$ with quotient U) and admits a free Hamiltonian $S^1(= S^1/(\mathbb{Z}/2))$ action with Hamiltonian $\bar{\mu} \colon U \to (0, \varepsilon)$.

Symplectic cutting U at $\varepsilon/2$ (see [17]) yields a symplectic manifold $\bar{N}$ diffeomorphic to the tubular neighborhood of the branch set $\bar{B} \subset S_g$. The symplectic structure on U is the restriction to U of the symplectic structure on $S_g - \bar{B}$ (pushed down from the equivariant symplectic structure on $F \times F - B$). Since symplectic cutting preserves the symplectic structure away from the cut locus it follows that S_g admits a symplectic form ω' whose restriction to $S_g - \bar{N}$ pulls back to the restriction of ω to $F \times F - N$.   □

Notice that the proof of Proposition 21 applies equally well to any regular branched cover $X \to Y = X/G$ with connected, symplectic branch manifold $B \subset X$ and G-equivariant symplectic form ω on X.

Proof of Theorem 20. Let F be a closed surface of genus g with a symplectic form ω_F. Let γ_1 and γ_2 be disjointly embedded curves in F representing different elements in a symplectic basis for $H_1(F)$. Then $T = \gamma_1 \times \gamma_2$ is a Lagrangian torus in $F \times F$. Since γ_1 and γ_2 are disjoint the composite $T \subset F \times F \to S_g$ is also an embedding.

Proposition 21 implies that this torus (which we continue to denote T) in S_g is Lagrangian with respect to a suitable symplectic form on S_g. The torus $T \subset S_g$ represents a non-trivial homology class in $H_2(S_g)$ since its transfer $\tau([F]) \in H_2(F \times F)$ is nonzero (it equals $\gamma_1 \times \gamma_2 + \gamma_1 \times \gamma_2$) by the Künneth theorem.

Thus the symplectic form on S_g can be perturbed slightly so that $T \subset S_g$ is symplectic. Taking the symplectic sum of S_g with the manifold K constructed in Lemma 18 so that yields a symplectic manifold M whose fundamental group is the quotient of $\pi_1(S_g) = \mathbb{Z}^{2g}$ by the subgroup generated by $[\gamma_1]$, i.e. $\pi_1(M) = \mathbb{Z}^{2g-1}$, and such that $\chi(M) = \chi(S_g) + 12$, $\sigma(M) = \sigma(S_g) - 8$. The calculations of Proposition 16 finish the proof. $\square$

5.4. Other abelian groups.

In Section 6 of [6] (Propositions 6.4 and 6.6), Gompf explores the geography of symplectic 4-manifolds with certain abelian fundamental groups constructed by symplectically summing torus bundles with $E(1)$. For completeness we state his results in our terminology.

Theorem 22 (Gompf). (1) *If G is the direct sum of up to three cyclic groups, except $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, or if $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/k \oplus \mathbb{Z}/\ell$ with $k, \ell \neq 0$, then there is a symplectic 4-manifold M with $\pi_1(M) = G$, $\chi(M) = 12$ and $\chi(M) + \sigma(M) = 4$.*

(2) *If G is $\mathbb{Z} \oplus \mathbb{Z}/k \oplus \mathbb{Z}/\ell \oplus \mathbb{Z}/n$, or if $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/k$ with $k, \ell, n \neq 0$, then there is a symplectic 4-manifold M with $\pi_1(M) = G$, $\chi(M) = 24$ and $\chi(M) + \sigma(M) = 8$.*

Note that these computations include the computations we gave for cyclic groups in the previous subsections. Using the same arguments as in the previous subsections, the first statement in Theorem 22 has the following consequences:

(1) If $G = \mathbb{Z}/k \oplus \mathbb{Z}/\ell \oplus \mathbb{Z}/n$ with $k, \ell, n \neq 0$, then

$$3 \leq \inf_{\mathfrak{M}(G)} \chi(M) \leq 12 \quad \text{and} \quad \inf_{\mathfrak{M}(G)} \chi(M) + \sigma(M) = 4.$$

(2) If $G = \mathbb{Z}/k \oplus \mathbb{Z}/\ell \oplus \mathbb{Z}$ with $k, \ell \neq 0$, then

$$2 \leq \inf_{\mathfrak{M}(G)} \chi(M) \leq 12 \quad \text{and} \quad \inf_{\mathfrak{M}(G)} \chi(M) + \sigma(M) = 4.$$

(3) If $G = \mathbb{Z}/k \oplus \mathbb{Z}^2$ with $k \neq 0$, then

$$0 \leq \inf_{\mathfrak{M}(G)} \chi(M) \leq 12 \quad \text{and} \quad \inf_{\mathfrak{M}(G)} \chi(M) + \sigma(M) = 0 \text{ or } 4.$$

Corresponding (but weaker) bounds can be derived from the second statement of Theorem 22.

Gompf also gives examples of relatively small symplectic 4-manifolds with other (non-abelian) fundamental groups. We refer the interested reader to his beautiful article [6].

6. Some final remarks

We end with a brief discussion about some difficult issues surrounding minimizers of χ. The 4-dimensional Poincaré conjecture can be rephrased by saying that any simply connected topological (resp. smooth) 4-manifold with minimal Euler characteristic is homeomorphic (resp. diffeomorphic) to the 4-sphere. In other words, if one minimizes the Euler characteristic χ on the class of simply connected 4-manifolds, the minimizer is unique. Freedman's theorem [5] proves the Poincaré conjecture for topological manifolds, and the smooth question is one of the outstanding problems in 4-dimensional topology.

New wrinkles appear in the symplectic case. For example $\mathbb{CP}^2$ minimizes the Euler characteristic among simply-connected symplectic 4-manifolds, and Freedman's theorem implies any two minimizers are homeomorphic. One might call the problem of whether any two simply connected symplectic 4-manifolds with $\chi = 3$ are diffeomorphic (or symplectomorphic) the "symplectic Poincaré conjecture". A counterexample would involve finding a simply-connected symplectic 4-manifold (M, ω) having $\chi(M) = 3$ and $K_M \cdot [\omega] > 0$ (cf. [18] or [20]). The question of whether a simply connected symplectic manifold with $\chi = 3$ is diffeomorphic or symplectomorphic to $\mathbb{CP}^2$ is unresolved, but there has been much recent progress in the direction of a counterexample. Starting with [25] and expanded upon in [24], [28], [4], [26], new examples were constructed of irreducible smooth 4-manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ for $n = 5, 6, 7, 8$. However, for $n = 5$ the examples are not symplectic.

All attempts to change the diffeomorphism type of known minimizers without changing their fundamental group seem to fail, suggesting that minimizers of $\chi : \mathfrak{M}(G) \to \mathbb{Z}$ are somehow special. But to conjecture that a symplectic minimizer of $\chi : \mathfrak{M}(G) \to \mathbb{Z}$ is unique up to diffeomorphism, however, is simply incorrect. For example, $S^2 \times T^2$ and the nontrivial S^2-bundle over T^2 both have fundamental group $\mathbb{Z}^2$. Yet the search for other examples with $G = \mathbb{Z}^2$ seems futile. It is certainly easy to build homology $T^2 \times S^2$ symplectic manifolds: let Y be zero surgery on a fibered knot in S^3 and take $Y \times S^1$. The only example from this extensive list that has fundamental group $\mathbb{Z}^2$ is when the knot is the unknot, i.e., when $Y \times S^1$ is diffeomorphic to $T^2 \times S^2$. The key difference between $S^2 \times T^2$ and the nontrivial S^2-bundle over T^2 is that the first is spin and the second is not. So minimizers of

$\chi : \mathfrak{M}(G) \to \mathbb{Z}$ can have different intersection forms of the same rank. This leads us to make, possibly out of ignorance, the following conjecture:

Conjecture 23. Let M be a symplectic 4-manifold with $\pi_1(M) \cong G$ which minimizes $\chi : \mathfrak{M}(G) \to \mathbb{Z}$. Let Q_M denote the intersection form of M. Then any other symplectic manifold with intersection form Q_M which also minimizes $\chi : \mathfrak{M}(G) \to \mathbb{Z}$ is diffeomorphic to M.

We offer this conjecture merely as a new twist on an old theme in 4-manifold theory, namely, describing conditions under which 4-manifolds are possibly unique. A weaker conjecture would be to let Q_M denote the equivariant (i.e. $\mathbb{Z}[G]$) intersection form of M. A counterexample to this conjecture would also be interesting. A good place to start is to find another minimizer of $\mathfrak{M}(\mathbb{Z}^6)$ which is not diffeomorphic to S_3. Notice that any minimizer of χ is necessarily minimal. If G is not a free product then any minimizer of χ is irreducible.

Suppose instead that one looks for minima of $\chi + \sigma$ on $\mathfrak{M}(e)$. Then minimizers are not unique: for example $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ are minimizers in $\mathfrak{M}(e)$. These examples indicate that to go beyond excessively general observations one may have to restrict further the class of manifolds, e.g. irreducible manifolds. Even then minimizers are not unique (up to diffeomorphism, for example). Indeed there are examples mentioned above of irreducible, symplectic 4-manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ for $n = 6$ (cf. [28]).

We end this article with remarks about improving our bounds.

What is missing in our results is a method for increasing the lower bounds of $\min_{M \in \mathfrak{M}(G)} \chi(M)$ which uses the fact that M is symplectic in a non-trivial way. The lower bounds given in the present article are obtained by combining the lower bounds valid for all 4-dimensional Poincaré complexes (e.g. Equation (1)) with two simple facts which hold for symplectic manifolds: $b^+(M) \geq 1$ and $1 - b_1(M) + b^+(M)$ is even. This second fact depends only the existence of an almost complex structure. Our calculations show that for $G = \mathbb{Z}^{2g}$, the difference

$$\min_{\mathfrak{M}(\mathbb{Z}^{2g})} \chi(M) - \min_{\mathfrak{M}^{\infty}(\mathbb{Z}^{2g})} \chi(M)$$

equals zero or one. On the other hand, a recent article of Kotschick [15] shows that for G_k the free group on k generators, the difference

$$\min_{\mathfrak{M}(G_k)} \chi(M) - \min_{\mathfrak{M}^{\infty}(G_k)} \chi(M)$$

gets arbitrarily large as k goes to infinity. Thus any improvement of the lower bounds which uses the symplectic structure in a deeper way will have to take these kinds of examples into account.

As explained at the end of Section 4, improving our upper bounds requires that we find a symplectic 4-manifold K with $\chi(K) < 12$ which contains a symplectically embedded torus T of self-intersection number zero with $\pi_1(K - T) \cong \mathbb{Z}$ or $\pi_1(K - T) = 1$. We have not found any such manifold, and might conjecture that one does not exist. It is not hard to show that any such K must satisfy $\chi(K) \geq 6$.

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