Complete constant mean curvature surfaces in homogeneous spaces

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Complete constant mean curvature surfaces in homogeneous spaces

José M. Espinar*and Harold Rosenberg

Abstract. In this paper we classify complete surfaces of constant mean curvature whose Gaussian curvature does not change sign in a simply connected homogeneous manifold with a 4-dimensional isometry group.

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1. Introduction

In 1966, T. Klotz and R. Ossermann showed the following:

Theorem ([KO]). *A complete H-surface in $\mathbb{R}^3$ whose Gaussian curvature K does not change sign is either a sphere, a minimal surface, or a right circular cylinder.*

The above result was extended to $\mathbb{S}^3$ by D. Hoffman [H], and to $\mathbb{H}^3$ by R. Tribuzy [T] with an extra hypothesis if K is non-positive. The additional hypothesis says that, when $K \leq 0$, one has $H^2 - K - 1 > 0$.

In recent years, the study of H-surfaces in product spaces and, more generally, in a homogeneous three-manifold with a 4-dimensional isometry group is quite active (see [AR], [AR2], [CoR], [ER], [FM], [FM2], [DH] and references therein).

The aim of this paper is to extend the above theorem to homogeneous spaces with a 4-dimensional isometry group. These homogeneous spaces are denoted by $\mathbb{E}(\kappa, \tau)$, where κ and τ are constant and $\kappa - 4\tau^2 \neq 0$. They can be classified as $\mathbb{M}^2(\kappa) \times \mathbb{R}$ if $\tau = 0$, with $\mathbb{M}^2(\kappa) = \mathbb{S}^2(\kappa)$ if $\kappa > 0$ ($\mathbb{S}^2(\kappa)$ the sphere of curvature κ), and $\mathbb{M}^2(\kappa) = \mathbb{H}^2(\kappa)$ if $\kappa < 0$ ($\mathbb{H}^2(\kappa)$ the hyperbolic plane of curvature κ). If τ is not equal to zero, $\mathbb{E}(\kappa, \tau)$ is a Berger sphere if $\kappa > 0$, a Heisenberg space if $\kappa = 0$ (of

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bundle curvature τ), and the universal cover of $\mathrm{PSL}(2,\mathbb{R})$ if $\kappa < 0$. Henceforth we will suppose κ is plus or minus one or zero.

The paper is organized as follows. In Section 2, we establish the definitions and necessary equations for an H-surface. We also state here two classification results for H-surfaces. We prove them in Section 5 and Section 6 for the sake of completeness.

Section 3 is devoted to the classification of H-surfaces with non-negative Gaussian curvature,

Theorem 3.1. *Let $\Sigma \subset \mathbb{E}(\kappa,\tau)$ be a complete H-surface with $K \geq 0$. Then, Σ is either a rotational sphere (in particular, $4H^2 + \kappa > 0$), or a complete vertical cylinder over a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$.*

In Section 4 we continue with the classification of H-surfaces with non-positive Gaussian curvature.

Theorem 4.1. *Let $\Sigma \subset \mathbb{E}(\kappa,\tau)$ be a complete H-surface with $K \leq 0$ and $H^2 + \tau^2 - |\kappa - 4\tau^2| > 0$. Then, Σ is a complete vertical cylinder over a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$.*

The above theorem is not true without the inequality; for example, any complete minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ that is not a vertical cylinder.

In the Appendix, we give a result, which we think is of independent interest, concerning differential operators on a Riemannian surface Σ of the form $\Delta + g$, acting on $C^2(\Sigma)$-functions, where Δ is the Laplacian with respect to the Riemannian metric on Σ and $g \in C^0(\Sigma)$.

2. The geometry of surfaces in homogeneous spaces

Henceforth $\mathbb{E}(\kappa,\tau)$ denotes a complete simply connected homogeneous three-manifold with 4-dimensional isometry group. Such a three-manifold can be classified in terms of a pair of real numbers (κ,τ) satisfying $\kappa - 4\tau^2 \neq 0$. In fact, these manifolds are Riemannian submersions over a complete simply-connected surface $\mathbb{M}^2(\kappa)$ of constant curvature κ, $\pi\colon \mathbb{E}(\kappa,\tau) \to \mathbb{M}^2(\kappa)$, and translations along the fibers are isometries, therefore they generate a Killing field ξ, called the *vertical field*. Moreover, τ is the real number such that $\overline{\nabla}_X \xi = \tau X \wedge \xi$ for all vector fields X on the manifold. Here, $\overline{\nabla}$ is the Levi-Civita connection of the manifold and $\wedge$ is the cross product.

Let Σ be a complete H-surface immersed in $\mathbb{E}(\kappa,\tau)$. By passing to a 2-sheeted covering space of Σ, we can assume Σ is orientable. Let N be a unit normal to Σ. In terms of a conformal parameter z of Σ, the first, $\langle\cdot,\cdot\rangle$, and second, II, fundamental forms are given by

$$\begin{aligned} \langle \cdot , \cdot \rangle &= \lambda \, |dz|^2 \\ II &= p \, dz^2 + \lambda \, H \, |dz|^2 + \bar{p} \, d\bar{z}^2, \end{aligned} \tag{2.1}$$

where $p \, dz^2 = \langle -\nabla_{\partial_z} N, \partial_z \rangle \, dz^2$ is the Hopf differential of Σ.

Set $\nu = \langle N, \xi \rangle$ and $T = \xi - \nu N$, i.e., ν is the normal component of the vertical field ξ, called the *angle function*, and T is the tangent component of the vertical field.

First we state the following necessary equations on Σ which were obtained in [FM].

Lemma 2.1. *Given an immersed surface $\Sigma \subset \mathbb{E}(\kappa, \tau)$, the following equations are satisfied:*

$$K = K_e + \tau^2 + (\kappa - 4\tau^2) \, \nu^2, \tag{2.2}$$

$$p_{\bar{z}} = \frac{\lambda}{2} \, (H_z + (\kappa - 4\tau^2) \, \nu \, A), \tag{2.3}$$

$$A_{\bar{z}} = \frac{\lambda}{2} \, (H + i \, \tau) \, \nu, \tag{2.4}$$

$$\nu_z = -(H - i \, \tau) \, A - \frac{2}{\lambda} \, p \, \bar{A}, \tag{2.5}$$

$$|A|^2 = \frac{1}{4} \, \lambda \, (1 - \nu^2), \tag{2.6}$$

$$A_z = \frac{\lambda_z}{\lambda} \, A + p \, \nu, \tag{2.7}$$

where $A = \langle \xi, \partial_z \rangle$, K_e the extrinsic curvature and K the Gauss curvature of Σ.

For an immersed H-surface $\Sigma \subset \mathbb{E}(\kappa, \tau)$ there is a globally defined quadratic differential, called the *Abresch–Rosenberg differential*, which in these coordinates is given by (see [AR2]):

$$Q \, dz^2 = (2(H + i \, \tau) \, p - (\kappa - 4\tau^2) A^2) \, dz^2,$$

following the notation above.

It is not hard to verify this quadratic differential is holomorphic on an H-surface using (2.3) and (2.4),

Theorem 2.1 ([AR], [AR2]). *$Q \, dz^2$ is a holomorphic quadratic differential on any H-surface in $\mathbb{E}(\kappa, \tau)$.*

Associated to the Abresch–Rosenberg differential we define the smooth function $q \colon \Sigma \to [0, +\infty)$ given by

$$q = \frac{4 |Q|^2}{\lambda^2}.$$

By means of Theorem 2.1, q either has isolated zeroes or vanishes identically. Note that q does not depend on the conformal parameter z, hence q is globally defined on Σ.

We continue this section establishing some formulae relating the angle function, q and the Gaussian curvature.

Lemma 2.2. *Let Σ be an H-surface immersed in $\mathbb{E}(\kappa,\tau)$. Then the following equations are satisfied:*

$$\begin{aligned}\|\nabla\nu\|^2 = {} & \frac{4H^2+\kappa-(\kappa-4\tau^2)\nu^2}{4(\kappa-4\tau^2)}(4(H^2-K_e) \\ & +(\kappa-4\tau^2)(1-\nu^2)) - \frac{q}{\kappa-4\tau^2},\end{aligned} \tag{2.8}$$

$$\Delta\nu = -\left(4H^2+2\tau^2+(\kappa-4\tau^2)(1-\nu^2)-2K_e\right)\nu. \tag{2.9}$$

Moreover, away from the isolated zeroes of q, we have

$$\Delta \ln q = 4K. \tag{2.10}$$

Proof. From (2.5)

$$|\nu_z|^2 = \frac{4\,|p|^2\,|A|^2}{\lambda^2} + (H^2+\tau^2)\,|A|^2 + \frac{2\,(H+i\tau)}{\lambda}p\bar{A}^2 + \frac{2\,(H-i\tau)}{\lambda}\bar{p}A^2,$$

and taking into account that

$$\begin{aligned}|Q|^2 = {} & 4\,(H^2+\tau^2)\,|p|^2 + (\kappa-4\tau^2)^2|A|^4 - (\kappa-4\tau^2)(2\,(H+i\tau)p\bar{A}^2 \\ & + 2\,(H-i\tau)\bar{p}A^2),\end{aligned}$$

we obtain, using also (2.6), that

$$\begin{aligned}|\nu_z|^2 = {} & (H^2+\tau^2)|A|^2 + (H^2-K_e)|A|^2 + (\kappa-4\tau^2)\frac{|A|^4}{\lambda} \\ & + 4\left(\frac{H^2+\tau^2}{\kappa-4\tau^2}\right)\frac{|p|^2}{\lambda} - \frac{|Q|^2}{(\kappa-4\tau^2)\lambda}\end{aligned}$$

where we have used that $4|p|^2 = \lambda^2(H^2-K_e)$ and $\kappa-4\tau^2 \neq 0$. Thus

$$\begin{aligned}\|\nabla\nu\|^2 = \frac{4}{\lambda}|\nu_z|^2 = {} & (2H^2-K_e+\tau^2)(1-\nu^2) + \frac{\kappa-4\tau^2}{4}(1-\nu^2)^2 \\ & + 4\left(\frac{H^2+\tau^2}{\kappa-4\tau^2}\right)(H^2-K_e) - \frac{q}{\kappa-4\tau^2},\end{aligned}$$

and finally, re-ordering in terms of H^2-K_e, we obtain the first expression.

Next, by differentiating (2.5) with respect to $\bar{z}$ and using (2.7), (2.4) and (2.3), one gets

$$\nu_{z\bar{z}} = -(\kappa - 4\tau^2)\,\nu\,|A|^2 - \frac{2}{\lambda}\,|p|^2\,\nu - \frac{H^2+\tau^2}{2}\,\lambda\,\nu.$$

Then, from (2.6),

$$\nu_{z\bar{z}} = -\frac{\lambda\,\nu}{4}\Big((\kappa - 4\tau^2)(1-\nu^2) + \frac{8\,|p|^2}{\lambda^2} + 2\,(H^2+\tau^2)\Big),$$

thus

$$\Delta\nu = \frac{4}{\lambda}\nu_{z\bar{z}} = -\Big((\kappa - 4\tau^2)(1-\nu^2) + 2(H^2 - K_e) + 2\,(H^2+\tau^2)\Big)\,\nu.$$

Finally,

$$\Delta \ln q = \Delta \ln \frac{4|Q|^2}{\lambda^2} = -2\Delta \ln\lambda = 4K,$$

where we have used that $Q\,dz^2$ is holomorphic and the expression of the Gaussian curvature in terms of a conformal parameter. □

Remark 2.1. Note that (2.9) is nothing but the Jacobi equation for the Jacobi field ν.

Next, we recall a definition in these homogeneous spaces.

Definition 2.1. We say that $\Sigma \subset \mathbb{E}(\kappa,\tau)$ is a vertical cylinder over α if $\Sigma = \pi^{-1}(\alpha)$, where α is a curve on $\mathbb{M}^2(\kappa)$.

It is not hard to verify that if α is a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$, then $\Sigma = \pi^{-1}(\alpha)$ is complete and has constant mean curvature H. Moreover, these cylinders are characterized by $\nu \equiv 0$.

We now state two results about the classification of H-surfaces. They will be used in Sections 3 and 4, but we prove them in Section 5 and Section 6 for the sake of clarity. The first one concerns H-surfaces for which the angle function is constant. However, we need to introduce a family of surfaces that appear in the classification.

Definition 2.2. Denote by $\mathcal{S}_{\kappa,\tau}$ a family of complete H-surfaces in $\mathbb{E}(\kappa,\tau)$, $\kappa < 0$, satisfying for any $\Sigma \in \mathcal{S}_{\kappa,\tau}$:

- $4H^2 + \kappa < 0$.
- q vanishes identically on $\Sigma \in \mathcal{S}_{\kappa,\tau}$, i.e., Σ is invariant by a one parameter family of isometries.
- $0 < \nu^2 < 1$ is constant along Σ.
- $K_e = -\tau^2$ and $K = (\kappa - 4\tau^2)\nu^2 < 0$ are constants along Σ.

An anonymous referee indicated to us the preprint "Hypersurfaces with a parallel higher fundamental form" by S. Verpoort who observed that we mistakenly omitted the surfaces $\mathcal{S}_{\kappa,\tau}$ in a first draft of this paper.

Theorem 2.2. *Let $\Sigma \subset \mathbb{E}(\kappa,\tau)$ be a complete H-surface with constant angle function. Then Σ is either a vertical cylinder over a complete curve of curvature $2H$ on $\mathbb{M}^2(\kappa)$, a slice in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$, or $\Sigma \in \mathcal{S}_{\kappa,\tau}$ with $\kappa < 0$.*

Remark 2.2. Theorem 2.2 improves Lemma 2.3 in [ER] for surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Of special interest for us are those H-surfaces for which the Abresch–Rosenberg differential is constant.

Theorem 2.3. *Let $\Sigma \subset \mathbb{E}(\kappa,\tau)$ be a complete H-surface with q constant.*

- *If $q = 0$, then Σ is invariant by a one-parameter group of isometries of $\mathbb{E}(\kappa,\tau)$, and if $H = 0 = \tau$, then Σ is a slice in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$.*

  Moreover, the Gauss curvature of these examples is as follows.

  - *If $4H^2 + \kappa > 0$, then $K = 0$, and they are rotationally invariant spheres.*
  - *If $4H^2 + \kappa = 0$ and $\nu \equiv 0$, then $K \equiv 0$ and Σ is either a vertical plane in Nil_3, or a vertical cylinder over a horocycle in $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{PSL}}(2,\mathbb{C})$.*
  - *There exists a point with negative Gauss curvature in the remaining cases.*

- *If $q \neq 0$ on Σ, then Σ is a vertical cylinder over a complete curve of curvature $2H$ on $\mathbb{M}^2(\kappa)$.*

3. Complete H-surfaces Σ with $K \geq 0$

Here we prove

Theorem 3.1. *Let $\Sigma \subset \mathbb{E}(\kappa,\tau)$ be a complete H-surface with $K \geq 0$. Then, Σ is either a rotational sphere (in particular, $4H^2 + \kappa > 0$), or a complete vertical cylinder over a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$.*

Proof. The proof goes as follows: First, we prove that Σ is a topological sphere or a complete non-compact parabolic surface. We show that when the surface is a topological sphere then it is a rotational sphere. If Σ is a complete non-compact parabolic surface, we prove that it is a vertical cylinder by means of Theorem 2.3.

Since $K \geq 0$ and Σ is complete, Lemma 5 in [KO] implies that Σ is either a sphere or non-compact and parabolic.

If Σ is a sphere, then it is a rotational example (see [AR2] or [AR]). Thus, we can assume that Σ is non-compact and parabolic.

We can assume that q does not vanish identically in Σ. If q does vanish, then Σ is either a vertical cylinder over a straight line in Nil_3 or a vertical cylinder over a horocycle in $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{PLS}(2,\mathbb{C})}$. Note that we have used here that $K \geq 0$ and Theorem 2.3.

On the one hand, from the Gauss equation (2.2)

$$0 \leq K = K_e + \tau^2 + (\kappa - 4\tau^2)\nu^2 \leq K_e + \tau^2 + |\kappa - 4\tau^2|,$$

hence

$$H^2 - K_e \leq H^2 + \tau^2 + |\kappa - 4\tau^2|. \tag{3.1}$$

On the other hand, using the very definition of $Q\,dz^2$, (3.1) and the inequality $|\xi_1 + \xi_2|^2 \leq 2(|\xi_1|^2 + |\xi|^2)$ for $\xi_1, \xi_2 \in \mathbb{C}$, we obtain

$$\begin{aligned}\frac{q}{2} = \frac{2|Q|^2}{\lambda^2} &\leq 4(H^2+\tau^2)\frac{4|p|^2}{\lambda^2} + (\kappa - 4\tau^2)^2\frac{4|A|^4}{\lambda^2}\\ &= 4(H^2+\tau^2)(H^2 - K_e) + \frac{(\kappa-4\tau^2)^2}{4}(1-\nu^2)^2\\ &\leq 4(H^2+\tau^2)(H^2 - K_e) + \frac{(\kappa-4\tau^2)^2}{4}\\ &\leq 4(H^2+\tau^2)(H^2 + \tau^2 + |\kappa - 4\tau^2|) + \frac{(\kappa-4\tau^2)^2}{4}.\end{aligned}$$

So, from (2.10), $\Delta \ln q = 4K \geq 0$ and $\ln q$ is a bounded subharmonic function on a non-compact parabolic surface Σ and since the value $-\infty$ is allowed at isolated points (see [AS]), q is a positive constant (recall that we are assuming that q does not vanish identically). Therefore, Theorem 2.3 gives the result. □

4. Complete H-surfaces Σ with $K \leq 0$

Theorem 4.1. *Let $\Sigma \subset \mathbb{E}(\kappa,\tau)$ be a complete H-surface with $K \leq 0$ and $H^2 + \tau^2 - |\kappa - 4\tau^2| > 0$. Then, Σ is a complete vertical cylinder over a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$.*

Proof. We divide the proof into two cases, $\kappa - 4\tau^2 < 0$ and $\kappa - 4\tau^2 > 0$.

Case $\kappa - 4\tau^2 < 0$: On the one hand, since $K \leq 0$, we have

$$H^2 - K_e \geq H^2 + \tau^2 + (\kappa - 4\tau^2)\nu^2 \geq H^2 + \kappa - 3\tau^2,$$

from the Gauss equation (2.2). Therefore, from (2.8) and $\kappa - 4\tau^2 < 0$, we obtain:

$$\begin{aligned} q &\geq 4(H^2+\tau^2)(H^2-K_e) + (\kappa-4\tau^2)(1-\nu^2) \\ &\quad \cdot \Big(H^2+\tau^2+H^2-K_e+\frac{\kappa-4\tau^2}{4}(1-\nu^2)\Big) \\ &= (H^2-K_e)\big(4H^2+4\tau^2+(\kappa-4\tau^2)(1-\nu^2)\big) \\ &\quad + (H^2+\tau^2)(\kappa-4\tau^2)(1-\nu^2) + \frac{(\kappa-4\tau^2)^2}{4}(1-\nu^2)^2 \\ &\geq (H^2+\tau^2+(\kappa-4\tau^2)\nu^2)\big(4H^2+4\tau^2+(\kappa-4\tau^2)(1-\nu^2)\big) \\ &\quad + (H^2+\tau^2)(\kappa-4\tau^2)(1-\nu^2) + \frac{(\kappa-4\tau^2)^2}{4}(1-\nu^2)^2; \end{aligned}$$

note that the last inequality holds since $4H^2+4\tau^2+(\kappa-4\tau^2)(1-\nu^2) \geq 4H^2+\kappa > 0$. $4H^2+\kappa > 0$ follows from

$$0 < 4(H^2+\tau^2) - |\kappa-4\tau^2| = 4H^2+\kappa.$$

Set $a := H^2+\tau^2$ and $b := \kappa-4\tau^2$. Define the real smooth function $f\colon [-1,1] \to \mathbb{R}$ as

$$f(x) = (a+bx^2)(4a+b(1-x^2)) + ab(1-x^2) + \frac{b^2}{4}(1-x^2)^2. \tag{4.1}$$

Note that $q \geq f(\nu)$ on Σ, $f(\nu)$ is just the last part in the above inequality involving q. It is easy to verify that the only critical point of f in $(-1,1)$ is $x=0$. Moreover,

$$f(0) = (4a+b)^2/4 > 0 \quad \text{and} \quad f(\pm 1) = 4a(a+b) > 0.$$

Actually, $f: \mathbb{R} \to \mathbb{R}$ has two others critical points, $x = \pm\sqrt{\frac{4a+b}{3|b|}}$, but here we have used that

$$\frac{4a+b}{3|b|} > 1,$$

since $0 < 4(H^2+\kappa-3\tau^2) = (4H^2+\kappa) - 3|\kappa-4\tau^2| = (4a+b) - 3|b|$.

So, set $c = \min\{f(0), f(\pm 1)\} > 0$, then

$$q \geq f(\nu) \geq c > 0.$$

Now, from (2.10) and $q \geq c > 0$ on Σ, it follows that $ds^2 = \sqrt{q}I$ is a complete flat metric on Σ and

$$\Delta^{ds^2} \ln q = \frac{1}{\sqrt{q}}\Delta \ln q = \frac{4K}{\sqrt{q}} \leq 0.$$

Since q is bounded below by a positive constant and (Σ, ds^2) is parabolic, then $\ln q$ is constant which implies that q is a positive constant. Thus, the result follows from Theorem 2.3. The case $\kappa - 4\tau^2 < 0$ is proved.

Case $\kappa - 4\tau^2 > 0$: Set $w_1 := 2(H + i\tau)\frac{p}{\lambda}$ and $w_2 := (\kappa - 4\tau^2)\frac{A^2}{\lambda}$, i.e., $q = 4|w_1 - w_2|^2$. Then

$$|w_1|^2 = (H^2 + \tau^2)(H^2 - K_e) \geq (H^2 + \tau^2)^2,$$

$$|w_2|^2 = \frac{(\kappa - 4\tau^2)^2}{16}(1 - \nu^2)^2 \leq \left(\frac{\kappa - 4\tau^2}{4}\right)^2,$$

where we have used that $H^2 - K_e \geq H^2 + \tau^2 + (\kappa - 4\tau^2)\nu^2 \geq H^2 + \tau^2$, since $K \leq 0$ and $\kappa - 4\tau^2 > 0$.

We recall a well-known inequality for complex numbers. Let $\xi_1, \xi_2 \in \mathbb{C}$, then $|\xi_1 + \xi_2|^2 \geq \big||\xi_1| - |\xi_2|\big|^2$. Thus,

$$\begin{aligned}\frac{1}{4}q &\geq \big||w_1| - |w_2|\big|^2 \geq \left|(H^2 + \tau^2) - \frac{|\kappa - 4\tau^2|}{4}\right|^2 \\ &= \frac{1}{16}\left|4(H^2 + \tau^2) - |\kappa - 4\tau^2|\right|^2 > 0.\end{aligned}$$

So, as q is bounded below by a positive constant, then, arguing as in the previous case, q is a constant. Thus, the result follows from Theorem 2.3. The case $\kappa - 4\tau^2 > 0$ is proved. □

Remark 4.1. Note that in the above theorem, in the case $\kappa - 4\tau^2 > 0$, we only need to assume that $4(H^2 + \tau^2) - |\kappa - 4\tau^2| > 0$.

5. Complete H-surfaces with constant angle function

We classify here the complete H-surfaces in $\mathbb{E}(\kappa, \tau)$ with constant angle function. The purpose is to take advantage of this classification result in the next section.

Theorem 2.2. *Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a complete H-surface with constant angle function. Then Σ is either a vertical cylinder over a complete curve of curvature $2H$ on $\mathbb{M}^2(\kappa)$, a slice in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$, or $\Sigma \in \mathcal{S}_{\kappa,\tau}$ with $\kappa < 0$ (see Definition 2.2).*

Proof. We can assume that $\nu \leq 0$. We will divide the proof into three cases:

- $\nu = 0$: In this case, Σ must be a vertical cylinder over a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$.

- $\nu = -1$: From (2.4), $\tau = 0$ and $H = 0$, then Σ is a slice in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$.
- $-1 < \nu < 0$: We prove here that $\Sigma \in \mathcal{S}_{\kappa,\tau}$ with $\kappa < 0$. From (2.5) we have

$$(H - i\tau)A = -\frac{2p}{\lambda}\bar{A}, \tag{5.1}$$

then

$$H^2 + \tau^2 = \frac{4|p|^2}{\lambda^2} = H^2 - K_e$$

since $|A|^2 \neq 0$ from (2.6), so $K_e = -\tau^2$ on Σ.

Thus, from (2.9), we have

$$4H^2 + 4\tau^2 + (\kappa - 4\tau^2)(1 - \nu^2) = 0. \tag{5.2}$$

Now, using the definition of q, (5.1), (5.2) and $K_e = -\tau^2$, we have

$$\begin{aligned} q = \frac{4|Q|^2}{\lambda^2} &= 4(H^2 + \tau^2)\frac{4|p|^2}{\lambda^2} + (\kappa - 4\tau^2)^2\frac{4|A|^4}{\lambda^2} \\ &\quad - 4\frac{\kappa - 4\tau^2}{\lambda^2}\left(2(H + i\tau)p\bar{A}^2 + 2(H - i\tau)\bar{p}A^2\right) \\ &= 4(H^2 + \tau^2)(H^2 - K_e) + (\kappa - 4\tau^2)^2\frac{(1 - \nu^2)^2}{4} \\ &\quad + 2(\kappa - 4\tau^2)(1 - \nu^2)(H^2 + \tau^2) \\ &= \frac{1}{4}\left(4H^2 + (\kappa - 4\tau^2)(1 - \nu^2) + 4\tau^2\right)^2 = 0, \end{aligned}$$

that is, q vanishes identically on Σ. Moreover, from (5.2), we can see that $4H^2 + \kappa < 0$, that is, $\kappa < 0$. Therefore, $\Sigma \in \mathcal{S}_{\kappa,\tau}$, $\kappa < 0$. □

6. Complete H-surfaces with q constant

Here, we prove the classification result for complete H-surfaces in $\mathbb{E}(\kappa, \tau)$ employed in the proof of Theorem 3.1 and Theorem 4.1.

Theorem 2.3. *Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a complete H-surface with q constant.*

- *If $q = 0$ on Σ, then Σ is either a slice in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$ if $H = 0 = \tau$, or Σ is invariant by a one-parameter group of isometries of $\mathbb{E}(\kappa, \tau)$.*

  Moreover, the Gauss curvature of these examples is as follows.

  - *If $4H^2 + \kappa > 0$, then $K > 0$ they are the rotationally invariant spheres.*

  - *If* $4H^2 + \kappa = 0$ *and* $\nu \equiv 0$*, then* $K \equiv 0$ *and* Σ *is either a vertical plane in* Nil_3*, or a vertical cylinder over a horocycle in* $\mathbb{H}^2 \times \mathbb{R}$ *or* $\widetilde{\mathrm{PSL}}(2, \mathbb{C})$.
  - *There exists a point with negative Gauss curvature in the remaining cases.*
- *If* $q \neq 0$ *on* Σ*, then* Σ *is a vertical cylinder over a complete curve of curvature* $2H$ *on* $\mathbb{M}^2(\kappa)$.

The case $q = 0$ has been treated extensively when the target manifold is a product space, but is has not been established explicitly when $\tau \neq 0$. So, we assemble the results in [AR], [AR2] for the reader's convenience.

Lemma 6.1. *Let* $\Sigma \subset \mathbb{E}(\kappa, \tau)$ *be a complete* H*-surface whose Abresch–Rosenberg differential vanishes. Then* Σ *is either a slice in* $\mathbb{H}^2 \times \mathbb{R}$ *or* $\mathbb{S}^2 \times \mathbb{R}$ *if* $H = 0 = \tau$*, or* Σ *is invariant by a one-parameter group of isometries of* $\mathbb{E}(\kappa, \tau)$.

Moreover, the Gauss curvature of these examples is as follows.

- *If* $4H^2 + \kappa > 0$*, then* $K > 0$ *they are the rotationally invariant spheres.*
- *If* $4H^2 + \kappa = 0$ *and* $\nu \equiv 0$*, then* $K \equiv 0$ *and* Σ *is either a vertical plane in* Nil_3*, or a vertical cylinder over a horocycle in* $\mathbb{H}^2 \times \mathbb{R}$ *or* $\widetilde{\mathrm{PSL}}(2, \mathbb{C})$.
- *There exists a point with negative Gauss curvature in the remaining cases.*

Proof. The idea of the proof for product spaces that we use below can be found in [dCF] and [FM].

If $H = 0 = \tau$, from the definition of the Abresch–Rosenberg differential, we have

$$0 = -(\kappa - 4\tau)A^2,$$

that is, $\nu^2 = \pm 1$ using (2.6). Thus, Σ is a slice in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$.

If $H \neq 0$ or $\tau \neq 0$, we have

$$2(H + i\tau)p = (\kappa - 4\tau^2)A^2, \tag{6.1}$$

from where we obtain, taking modulus,

$$H^2 - K_e = \frac{(\kappa - 4\tau^2)^2(1 - \nu^2)^2}{16(H^2 + \tau^2)}. \tag{6.2}$$

Inserting (6.1) in (2.5),

$$(H + i\tau)\nu_z = -\frac{1}{4}(4H^2 + \kappa - (\kappa - 4\tau^2)\nu^2)A,$$

and taking modulus,

$$|\nu_z|^2 = g(\nu)^2|A|^2, \quad g(\nu) = \frac{4H^2 + \kappa - (\kappa - 4\tau^2)\nu^2}{4\sqrt{H^2 + \tau^2}}. \tag{6.3}$$

Assume that ν is not constant. Let $p \in \Sigma$ be a point where $\nu_z(p) \neq 0$ and let $\mathcal{U}$ be a neighborhood of that point p where $\nu_z \neq 0$ (we can assume $\nu^2 \neq 1$ at p). In particular, $g(\nu) \neq 0$ in $\mathcal{U}$ from (6.3). Now, inserting (6.3) in (2.6), we obtain

$$\lambda = \frac{4|\nu_z|^2}{(1-\nu^2)g(\nu)^2}. \tag{6.4}$$

Thus, putting (6.2) and (6.4) in the Jacobi equation (2.9)

$$\nu_{z\bar{z}} = -2\frac{\nu|\nu_z|^2}{1-\nu^2}. \tag{6.5}$$

So, define the real function $s := \operatorname{arctgh}(\nu)$ on $\mathcal{U}$. Such a function is harmonic by means of (6.5), thus we can consider a new conformal parameter w for the first fundamental form so that $s = \operatorname{Re}(w)$, $w = s + it$.

Since $\nu = \operatorname{tgh}(s)$ by the definition of s, we have that $\nu \equiv \nu(s)$, i.e., it only depends on one parameter. Thus, we have $\lambda \equiv \lambda(s)$ and $T \equiv T(s)$ from (6.4) and (6.3) respectively, and $p \equiv p(s)$ by the definition of the Abresch–Rosenberg differential. That is, all the fundamental data of Σ depend only on s.

Now, let $\mathcal{U}$ be a simply connected domain on Σ and $\mathcal{V} \subset \mathbb{R}^2$ a simply connected domain of a surface S so that $\psi_0 \colon \mathcal{V} \to \mathcal{U} \subset \mathbb{E}(\kappa,\tau)$. We parametrize $\mathcal{V}$ by the parameters (s,t) obtained above. Then, the fundamental data (see [FM], Theorem 2.3) $\{\lambda_0, p_0, T_0, \nu_0\}$ of ψ_0 are given by

$$\begin{cases} \lambda_0(s,t) = \lambda(s), \\ p_0(s,t) = p(s), \\ T_0(s,t) = a(s)\partial_s, \\ \nu_0(s,t) = \nu(s), \end{cases}$$

where $a(s)$ is a smooth function.

Let $\bar{t} \in \mathbb{R}$ and let $\mathbf{i}_{\bar{t}} \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the diffeomorphism given by

$$\mathbf{i}_{\bar{t}}(s,t) := (s, t+\bar{t}),$$

and define $\psi_{\bar{t}} := \psi_0 \circ \mathbf{i}_{\bar{t}}$. Then, the fundamental data $\{\lambda_{\bar{t}}, p_{\bar{t}}, T_{\bar{t}}, \nu_{\bar{t}}\}$ of $\psi_{\bar{t}}$ are given by

$$\begin{cases} \lambda_{\bar{t}}(s,t) = \lambda(s), \\ p_{\bar{t}}(s,t) = p(s), \\ T_{\bar{t}}(s,t) = a(s)\partial_s, \\ \nu_{\bar{t}}(s,t) = \nu(s), \end{cases}$$

that is, both fundamental data match at any point $(s,t) \in \mathcal{V}$. Therefore, using [D], Theorem 4.3, there exists an ambient isometry $\mathcal{I}_{\bar{t}} \colon \mathbb{E}(\kappa,\tau) \to \mathbb{E}(\kappa,\tau)$ so that

$$\mathcal{I}_{\bar{t}} \circ \psi_0 = \psi_0 \circ \mathbf{i}_{\bar{t}} \quad \text{for all } \bar{t} \in \mathbb{R},$$

thus the surface is invariant by a one parameter group of isometries.

Let us prove the claim about the Gauss curvature. Using the Gauss equation (2.2) in (6.2), one gets

$$H^2+\tau^2+(\kappa-4\tau^2)\nu^2-K=\frac{(\kappa-4\tau^2)^2(1-\nu^2)^2}{16(H^2+\tau^2)}.$$

Set $a:=4(H^2+\tau^2)$ and $b:=\kappa-4\tau^2$, then one can check easily that the above equality can be expressed as

$$4aK=a^2-b^2+(2a+b)^2-(2a+b(1-\nu^2))^2. \tag{6.6}$$

So, if $4H^2+\kappa>0$ then $a>|b|$ and $K>0$, that is, Σ is a topological sphere since it is complete. If $4H^2+\kappa=0$, $a=-b$ and the equation reads as

$$4aK=a^2(1-(1+\nu^2)^2),$$

that is, Σ has a point with negative Gauss curvature unless $\nu\equiv 0$.

If $4H^2+\kappa<0$, one can check that $a^2-b^2=(a-b)(a+b)<0$ since $a+b>0$ and $a-b<0$. So, if $\inf_\Sigma\{\nu^2\}=0$ then, from (6.6), Σ has a point with negative curvature. Therefore, to finish this lemma, we shall prove the following

Claim. There are no complete constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$ with $4H^2+\kappa<0$, $q\equiv 0$, $K\geq 0$, and $\inf\{\nu^2\}=c>0$.

Proof of the Claim. Assume such a surface Σ exists. Since we are assuming that $K\geq 0$ and Σ is complete, then Σ is parabolic and noncompact. If Σ were compact we would have a contradiction with the fact that $\inf_\Sigma\{\nu^2\}=c>0$ and $4H^2+\kappa<0$.

Since q vanishes identically on Σ, $\operatorname{arctanh}(\nu)$ is a bounded harmonic function on Σ and so ν is constant. So, the projection $\pi:\Sigma\to\mathbb{M}^2(\kappa)$ is a global diffeomorphism and a quasi-isometry. This is impossible since Σ is parabolic and $\mathbb{M}^2(\kappa)$, $\kappa<0$, is hyperbolic. Therefore, the Claim is proved and so the lemma is proved. □

Proof of Theorem 2.3. We focus on the case $q\neq 0$ because Lemma 6.1 gives the classification when $q=0$.

Suppose ν is not constant in Σ. Since $q=c^2>0$, we can consider a conformal parameter z so that $\langle\cdot,\cdot\rangle=|dz|^2$ and $Q\,dz^2=c\,dz^2$ on Σ. Thus,

$$Q=c=2(H+i\tau)p-(\kappa-4\tau^2)A^2.$$

First, note that we can assume that $H\neq 0$ or $\tau\neq 0$, otherwise ν would be constant. So, from (2.5), we have

$$(H+i\tau)\nu_z=-\Big(H^2+\tau^2+\frac{\kappa-4\tau^2}{4}(1-\nu^2)\Big)A-c\bar{A},$$

where we have used $2(H + i\tau)p = c + (\kappa - 4\tau^2)A^2$. That is,

$$16(H^2 + \tau^2)\,\|\nabla\nu\|^2 = (g(\nu) + 4c)^2\,(1 - \nu^2), \tag{6.7}$$

where

$$g(\nu) := 4H^2 + \kappa - (\kappa - 4\tau^2)\nu^2. \tag{6.8}$$

From (2.10), Σ is flat and $H^2 - K_e = H^2 + \tau^2 + (\kappa - 4\tau^2)\nu^2$ by (2.2), joining this last equation to (2.8) we obtain using the definition of $g(\nu)$ given in (6.8)

$$\|\nabla\nu\|^2 = \frac{g(\nu)^2}{4(\kappa - 4\tau^2)} + \nu^2 g(\nu) - \frac{c^2}{\kappa - 4\tau^2}. \tag{6.9}$$

Putting together (6.7) and (6.9) we obtain a polynomial expression in ν^2 with coefficients depending on $a := 4(H^2 + \tau^2)$, $b := \kappa - 4\tau^2$ and c:

$$P(\nu^2) := C(a,b,c)\nu^6 + \text{ lower terms } = 0,$$

but one can easily check that the coefficient of ν^6 is $C(a,b,c) = -a^{-1}b^2 \neq 0$, a contradiction. Thus ν is constant, and so, by means of Theorem 2.2, Σ is a vertical cylinder over a complete curve of curvature $2H$. □

7. Appendix

Let Σ be a connected Riemannian surface. We establish in this Appendix a result which we think is of independent interest, concerning differential operators of the form $\Delta + g$, acting on $C^2(\Sigma)$-functions, where Δ is the Laplacian with respect to the Riemannian metric on Σ and $g \in C^0(\Sigma)$.

Lemma 7.1. *Let $g \in C^0(\Sigma)$, $v \in C^2(\Sigma)$ such that $\|\nabla v\|^2 \leq h\,v^2$ on Σ, h is a non-negative continuous function on Σ, and $\Delta v + g v = 0$ in Σ. Then either v never vanishes or v vanishes identically on Σ.*

Proof. Set $\Omega = \{p \in \Sigma : v(p) = 0\}$. We will show that either $\Omega = \emptyset$ or $\Omega = \Sigma$.

So, let us assume that $\Omega \neq \emptyset$. If we prove that Ω is an open set then, since Ω is closed and Σ is connected, $\Omega = \Sigma$. Let $p \in \Omega$ and $\mathcal{B}(R) \subset \Sigma$ be the geodesic ball centered at p of radius R. Such a geodesic ball is relatively compact in Σ.

Set $\phi = v^2/2 \geq 0$. Then

$$\Delta\phi = v\Delta v + \|\nabla v\|^2 = -g v^2 + \|\nabla v\|^2 \leq -2(g - h)\phi,$$

that is,

$$-\Delta\phi - 2(g - h)\phi \geq 0. \tag{7.1}$$

Define $\beta := \min\{\inf_\Omega\{2(g-h)\}, 0\} \leq 0$. Then, $\psi = -\phi$ satisfies

$$\Delta\psi + \beta\psi = -\Delta\phi - \beta\phi \geq -\Delta\phi - 2(g-h)\phi \geq 0,$$

where we have used (7.1).

Since we are assuming that v has a zero at an interior point of $\mathcal{B}(R)$, $\beta \leq 0$ and ψ has a non-negative maximum at p, the Maximum Principle [GT], Theorem 3.5, implies that ψ is constant and so v is constant as well, i.e, $v \equiv 0$ in $\mathcal{B}(R)$. Then $\mathcal{B}(R) \subset \Omega$, and Ω is an open set. Thus $\Omega = \Sigma$. □

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José M. Espinar, Departamento de Geometría y Topología, Universidad de Granada, Avda Fuentenueva s/n, 18071 Granada, Spain

E-mail: jespinar@ugr.es

Harold Rosenberg, Instituto de Matematica Pura y Aplicada, 110 Estrada Dona Castorina, Rio de Janeiro 22460-320, Brazil

E-mail: rosen@impa.br