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Lower bounds for Pythagoras numbers of function fields

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Abstract. We show that the transcendence degree of a real function field over an arbitrary real base field is a strict lower bound for its Pythagoras number and a weak lower bound for all its higher Pythagoras numbers.

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1. Introduction

The *Pythagoras number* $p(F)$ of a field F is the smallest integer n such that every sum of squares in F is equal to a sum of n squares if such an n exists, or infinite otherwise. Recall that F is called *real* if -1 is not a sum of squares in F. The main result of this article is the following.

Theorem 1.1. *Let F be a real field that is finitely generated of transcendence degree d over a subfield K. Then $p(F) \geq d + 1$.*

For $K = \mathbb{R}$, this was shown by Kucharz in [11], and later extended in [12] to the case where K is a real closed field. Kucharz obtains the lower bound $d + 1$ by showing that it is the minimal number of generators of some finitely generated ideal in the real holomorphy ring of the function field. He uses the geometric description of real holomorphy rings by [5], which rely on Hironaka's resolution of singularities and points of indeterminacy of rational maps.

The present paper also uses geometric methods, in particular a version of Bertini's theorem, as well as valuation theoretic methods, and a generalization of a well known result of Cassels, Ellison and Pfister. However, in difference to Kucharz' proofs in [11] and [12], the results given here do not rely on Hironaka's resolution of singularities or points of indeterminacy.

The paper is structured as follows. In Section 2, we show the existence of certain valuations on a real function field F/K in d variables, in particular a valuation whose residue field is a rational function field in $d - 1$ variables over a finite real extension of K.

In Section 3, we first observe that the result of Cassels, Ellison and Pfister in [6] implies Theorem 1.1 with the better bound $p(F) \geq d+2$ in the case where F/K is a rational function field in $d \geq 2$ variables.

Then we consider the case of a general real function field F/K in $d \geq 3$ variables and show that the valuation obtained in Section 2 allows a reduction to the previous case of rational function fields. The case $d = 2$ is shown in a similar but slightly different way. The cases $d = 0, 1$ were known before.

We also show the weaker lower bound d for the so called *higher Pythagoras numbers* in Section 3. In contrast to the proof of the stronger lower bound $d+1$ for the usual Pythagoras number, the proof of the weaker bound for the higher Pythagoras number does rely on Kucharz' result, and therefore on resolution of singularities.

2. Some arithmetic of real function fields

We start with some general observations on valuation rings that are centered in a regular local ring or in a prime ideal of a polynomial ring.

Lemma 2.1. *Let $(R, \mathfrak{m})$ be a regular local domain with field of fractions F. Then there exists a valuation ring $(\mathcal{O}, \mathfrak{M})$ in F dominating $(R, \mathfrak{m})$ such that the natural embedding $R/\mathfrak{m} \hookrightarrow \mathcal{O}/\mathfrak{M}$ is an isomorphism.*

Proof. This follows for example from [1, Chapter II, Lemma 3.4]. □

Corollary 2.2. *Let K be a field and $\mathfrak{p}$ a prime ideal of the polynomial ring $K[X_1, \dots, X_d]$ for some $d \in N$. Then there exists a valuation ring $(\mathcal{O}, \mathfrak{m})$ with field of fractions $K(X_1, \dots, X_d)$ containing $K[X_1, \dots, X_d]$ with $\mathfrak{p} = \mathfrak{m} \cap K[X_1, \dots, X_d]$ and such that the field of fractions of $K[X_1, \dots, X_d]/\mathfrak{p}$ is canonically isomorphic to $\mathcal{O}/\mathfrak{m}$.*

Proof. Let $\mathcal{O}'$ denote the localization of $K[X_1, \dots, X_d]$ at $\mathfrak{p}$. We denote $\mathfrak{m}'$ the maximal ideal of $\mathcal{O}'$. Obviously $\mathcal{O}'/\mathfrak{m}'$ is canonically isomorphic to the fraction field of $K[X_1, \dots, X_d]/\mathfrak{p}$. As a localization of a polynomial ring over a field, we have ([7, Exercise 13.6]) that $\mathcal{O}'$ is a regular local ring. The statement now follows from Lemma 2.1. □

We recall from [4, Theorem 1.1.8] that real fields are exactly the fields that admit an ordering that is compatible with addition and multiplication. By a variety, we mean a reduced scheme of finite type over a field. We call a point on the variety *real* if its residue field is real. We call an irreducible variety *real* if its generic point (i.e. its function field) is real. The following is a well known fact to real geometers, at least in the situation of a real closed base field.

Proposition 2.3. *A irreducible variety is real if and only if it admits a nonsingular real point.*

Proof. Let V be an irreducible variety over a field K, and let F denote its function field. We can assume that V is affine and that K is real. Let us first assume that V contains a real nonsingular closed point P. In particular its local ring $\mathcal{O}_{V,P}$ is regular and its residue field $\kappa(P)$ is real. By Lemma 2.1, there exists a valuation v whose residue field is isomorphic to $\kappa(P)$.

Suppose F is nonreal. Then $0 = x_1^2 + \cdots + x_s^2$ for some $s \geq 1$ and $x_i \in F^\times$. Assume that $v(x_1) = \min\{v(x_1), \ldots, v(x_s)\}$. After dividing the equality by x_1^2, we can assume that $x_1 = 1$ and $v(x_i) \geq 0$ for all $1 \leq i \leq s$. It follows that -1 is a sum of squares in the residue field $\kappa(P)$ of v, which yields the contradiction. Hence F is real.

To show the converse implication, we start by assuming that F is real. In the case where K is real closed, [4, Proposition 7.6.4 (i)] yields the existence of a real nonsingular closed point in V. When K is an arbitrary real field, let K' denote the relative algebraic closure of K in F. Considered as a variety over K', we have that V is geometrically irreducible. Now let R denote a real closure of K' with respect to an ordering that extends to F. The base change V_R of V from base K' to base R is irreducible and its function field F_R is a real function field over R. By what we said earlier, there exists a nonsingular rational point $P_R \in V_R$ with $\kappa(P_R) = R$. Let P denote the image of P_R under the base change morphism $V_R \to V$. Then $\kappa(P) \subseteq \kappa(P_R) = R$, hence $P \in V$ is a nonsingular closed real point. □

Corollary 2.4. *The existence of a nonsingular real point on an irreducible variety is a birational invariant. In particular, the set of nonsingular closed real points on a real variety is Zariski dense.*

Another consequence of Proposition 2.3 is the following.

Proposition 2.5. *Let F/K be a real function field in $d \geq 1$ variables. Then F admits a discrete K-valuation of rank one whose residue field is a rational function field in $d-1$ variables over a finite real extension of K.*

Proof. Let V be a variety over K with function field F. By Proposition 2.3, V contains a nonsingular closed real point P.

The exceptional fiber of the blowing-up $\mathcal{B}\ell_{\{P\}}(V) : V' \to V$ of V along $\{P\}$ is isomorphic over K to $\mathbb{P}^{n-1}_{\kappa(P)}$ (see [14, Chap. 8, Thm. 1.19]). In particular, its generic point $\eta \in V'$ is of codimension one in V' with residue field $\kappa(\eta) \cong \kappa(P)(X_1, \ldots, X_{d-1})$. As V is regular in a neighborhood of P (see [14, Chap. 8, Corollary 2.40]), and so is V' in a neighborhood of η (by [14, Chap. 8, Thm. 1.19]). The local ring of V' at η is thus a discrete valuation ring, which yields the claimed discrete K-valuation of rank one on F. □

Remark 2.6. In the preprint version [8] of this article, a more precise version of the proposition is deduced from a general technical result [8, Lemma 4.1] from which also the following more general result follows easily: Let F be the field of fractions of a regular local ring of Krull dimension $d \geq 2$ and let E/K be a finite field extension. Then F admits a discrete valuation of rank one with residue field $E(X_1, \dots, X_{d-1})$.

In the following, we will show the existence of valuations on function fields with certain nonreal residue fields that will later be used to show the lower bound 3 for its Pythagoras numbers in the two-dimensional case.

Proposition 2.7. *Let K be a real field and V a geometrically irreducible projective K-variety of dimension at least 2. Then there exist infinitely many nonreal geometrically irreducible K-closed subsets $C \subseteq V$ of codimension one.*

Proof. Let $V \subset \mathbb{P}^n$ be a closed K-immersion in projective space. Let $\mathcal{Q}$ denote the variety of all quadric hypersurfaces of $\mathbb{P}^n$, and $\mathcal{Q}(K)$ the set of quadrics defined over K. Consider the degree-2 Veronese embedding

$$
\begin{aligned}
f : \mathbb{P}^n &\hookrightarrow \mathbb{P}^N \\
[x_0 : \cdots : x_n] &\mapsto [x_i x_j]_{0 \leq i,j \leq n}
\end{aligned}
$$

where $\mathbb{P}^N = Proj(\mathbb{Q}[Y_{i,j} \mid 0 \leq i \leq j \leq n])$. Note that $\mathcal{Q} = f^{-1}(\mathrm{Gr}(1, N))$, where $\mathrm{Gr}(1, N)$ is the grassmanian variety of linear subspaces of codimension one in $\mathbb{P}^N$. By a version of Bertini's theorem [10, Corollary 6.11] applied to $f|_V$ there exists a nonempty K-Zariski-open subset $\mathcal{U} \subseteq \mathcal{Q}(K)$ such that the K-variety $V \cap Q$ is geometrically irreducible for every $Q \in \mathcal{U}$. We claim that infinitely many quadrics $Q \in \mathcal{U}$ have a underlying quadratic form that is totally definite, i.e. definite at every field ordering of K. For such a quadric Q, we have that the geometrically irreducible subvariety $C := V \cap Q$ has no L-point over any real field extension L/K, and is thus nonreal.

In order to verify the existence of infinitely many quadrics $Q \in \mathcal{U}$ with totally definite underlying quadratic form, we start with the totally definite quadratic form $\varphi := \sum_{i=0}^n X_i^2$ defined over $\mathbb{Q}$. The quadric it defines may not be contained in U, but we can consider a neighborhood $\mathcal{W}_{\varphi,\varepsilon} \subseteq \mathcal{Q}(\mathbb{Q})$ in the real topology for a given positive $\epsilon \in \mathbb{Q}$ consisting of the quadrics defined by the quadratic forms $\sum_{0 \leq i \leq j \leq n} a_{i,j} X_i X_j$ with $a_{0,0} = 1$ and $a_{i,j} \in \mathbb{Q}$ with $|a_{i,j} - \delta_{i,j}| < \varepsilon$. Note that for ε small enough, all quadrics in $\mathcal{W}_{\varphi,\varepsilon}$ are given by totally definite quadratic forms. Since $\mathcal{W}_{\varphi,\varepsilon}$ is Zariski-dense in $\mathcal{Q}(K)$, we have that $\mathcal{W}_{\varphi,\varepsilon} \cap \mathcal{U} \neq \emptyset$, in fact this intersection is infinite, since we can replace $\mathcal{U}$ by any cofinite subset of $\mathcal{U}$ in the previous argument. □

Corollary 2.8. *Let K be a real field and F/K a function field in $d \geq 2$ variables such that K is relatively algebraically closed in F. There are infinitely many discrete K-valuations of rank one on F whose residue fields are nonreal and contain no proper algebraic extension of K.*

3. Lower bounds for Pythagoras numbers

As mentioned in the introduction, we know that $p(K(X_1, \ldots, X_d)) \geq d + 2$ when $d \geq 2$ for any real field K. More precisely, in [6] it was shown that the sum of 4 squares

$$\mathcal{M}(X, Y) = (1+X^2-2X^2Y^2)^2+(XY^2-X^3Y^2)^2+(XY-X^3Y)^2+(X^2Y-X^4Y)^2$$

in $\mathbb{Q}(X, Y)$ is not a sum of 3 squares in $\mathbb{R}(X, Y)$. Since the first order theory of real closed fields is model-complete (see e.g. [4, Proposition 5.2.3]), it follows that $\mathcal{M}(X, Y)$ is not a sum of 3 squares in $\mathcal{R}(X, Y)$ for any real closed field $\mathcal{R}$. This implies that $\mathcal{M}(X, Y)$ is not a sum of 3 squares in $K(X, Y)$ for any real field K, and hence $p(K(X, Y)) \geq 4$. An iteration argument based on the Cassels–Pfister representation theorem for quadratic forms [16, Chap. 1, Thm. 3.2] shows more generally for $d \geq 2$ that the sum of $d + 2$ squares

$$\mathcal{M}(X_1, X_2) + X_3^2 + \cdots + X_d^2$$

is not a sum of $d + 1$ squares in $K(X_1, \ldots, X_d)$ for any real field K. In particular $p(K(X_1, \ldots, X_d)) \geq d + 2$ when $d \geq 2$.

For a field F and a positive integer m, the $2m$-*th Pythagoras number of* F is

$$p_{2m}(F) = \inf \left\{ n \in \mathbb{N} \;\middle|\; \forall x \in F^{n+1} \, \exists y \in F^n \text{ with } \sum_{i=1}^{n+1} x_i^{2m} = \sum_{i=1}^{n} y_i^{2m} \right\},$$

which generalizes the definition of the Pythagoras number (i.e. $p(F) = p_2(F)$).

Remark 3.1. This definition could be extended to include the cases of powers of odd exponents $2m + 1$, but their study in context of function fields is not so interesting, as they are bounded from above by a constant (that only depends on m but not on the transcendence degree of the function field), as was observed in [3, Proposition 2.8]. Hence, when we speak of *higher Pythagoras numbers*, we only refer to the ones defined with respect to powers of even exponent $2m$ as above.

Becker showed in [3] that all higher Pythagoras numbers of a finitely generated field extension of $\mathbb{R}$ are finite. In fact, he found an effective bound depending only on the transcendence degree d of the extension and the 'order' m of the considered higher Pythagoras number p_{2m}.

While the optimality of Pfister's upper bound 2^d for the usual Pythagoras number ($m = 1$) is a big open question, Becker's bound is known not to be optimal. For example, it was shown in [18] that $p_4(\mathbb{R}(X)) \leq 6$, i.e. the 4th Pythagoras number is significantly smaller than Becker's lower bound, which is 36 in the one-dimensional case, and in [19] the upper bound 6 was verified[1] for all one-dimensional real function fields over $\mathbb{R}$.

In [12, Corollary 2] every higher Pythagoras number of a real field that is finitely generated of transcendence degree d over a real closed field was shown to be bounded from below by $d + 1$. This is also the first nontrivial general lower bound obtained for the usual Pythagoras number ($m = 1$). In this section, we will show the following.

Theorem 3.2 (Main result). *Let m be a positive integer. Let F/K be a real function field of transcendence degree d. Then $p_{2m}(F) \geq d$. Moreover, we have $p_{2m}(F) \geq d + 1$ when $m = 1$ or $d \leq 2$.*

The bound $p_2(F) \geq d + 1$ for $d \geq 3$ will be shown in Corollary 3.7. The weaker bound $p_{2m}(F) \geq d$ for all higher Pythagoras numbers will be shown in Corollary 3.10. Finally $p_{2m}(F) \geq d + 1$ when $d \leq 2$ will be shown in Proposition 3.12. □

As mentioned in the introduction, we first give a relative bound of $p_2(F)$ (resp. $p_{2m}(F)$) in terms of the Pythagoras number (resp. $2m$-th Pythagoras number) of a certain rational function field in $d - 1$ variables. For $d \geq 3$, in order to obtain the absolute bound $p_2(F) \geq d+1$ (resp. $p_{2m}(F) \geq d$), we show that $d+1$ (resp. d) is a lower bound for the Pythagoras number (resp. for the $2m$-th Pythagoras number) of the rational function field, by applying the previous mentioned quadratic form theoretic results for the Pythagoras number (resp. by considering a generic sum of d powers of exponent $2m$ in the rational function field, and reducing this case to Kucharz result). Let me stress that for obtaining the lower bound $p_2(F) \geq d + 1$ for the ususal Pythagoras number, we will not reduce to Kucharz result in any way, and even more, that there probably is no way to obtain such a reduction in general.

Analogously to the $2m$-th Pythagoras number, we denote the *$2m$-th level* of a field F for some positive integer m by

$$s_{2m}(F) = \inf\{n \in \mathbb{N} \mid -1 = f_1^{2m} + \cdots + f_n^{2m} \text{ for some } f_1, \ldots, f_n \in F\}.$$

We recall from [3, Thm.2.9] that $s_{2m}(F) < \infty$ if and only if F is nonreal.

Lemma 3.3. *Let m be a positive integer. Let v be a valuation on a field F with real residue field κ_v. Let $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \kappa_v$ such that $x_1^{2m} + \cdots + x_n^{2m} \neq y_1^{2m} + \cdots + y_{n-1}^{2m}$ for any $y_1, \ldots, y_{n-1} \in \kappa_v$. For $1 \leq i \leq n$ let $X_i \in F$ be a lift of $x_i \in \kappa_v$ with respect to the residue map. Then $X_1^{2m}+\cdots+X_n^{2m} \neq Z_1^{2m}+\cdots+Z_{n-1}^{2m}$ for any $Z_1, \ldots, Z_{n-1} \in F$.*

[1] A substantial gap in the proof pointed out by the author of [19] could be closed recently, as I learned from oral communication with E. Becker.

Proof. Assume that $X_1^{2m} + \cdots + X_n^{2m} = Z_1^{2m} + \cdots + Z_{n-1}^{2m}$ for some $Z_i \in F$. Since κ_v is real, it follows that $0 = v(X_1^{2m} + \cdots + X_n^{2m}) = 2m \min v(Z_i)$. In particular, we can apply the residue map to the Z_i and obtain the contradiction that $x_1^{2m} + \cdots + x_n^{2m} = z_1^{2m} + \cdots + z_{n-1}^{2m}$ in κ_v. □

We recall briefly that a valuation $v : F^\times \to \Gamma$ is called *discrete* if $v(F^\times)$ is discrete as an ordered abelian group. By the *rank of* v we denote the $\mathbb{Q}$-dimension of $v(F^\times) \otimes_{\mathbb{Z}} \mathbb{Q}$. One can show that the value group of a discrete valuation of finite rank r is order-isomorphic to $\mathbb{Z}^r$ endowed with lexicographic ordering.

Lemma 3.4. *Let m be a positive integer. Let v be a discrete valuation of finite rank on a field F with nonreal residue field κ_v of characteristic relatively prime to $2m$. Let $s \in \mathbb{N}$ be minimal such that there exist $x_1, \ldots, x_s \in \kappa_v$ with $-1 = x_1^{2m} + \cdots + x_s^{2m}$. For $1 \le i \le s$ let $X_i \in F$ be a lift of $x_i \in \kappa_v$ with respect to the residue map. Then there exists a lift $X_{s+1} \in F$ of $1 \in \kappa_v$ such that $X_1^{2m} + \cdots + X_{s+1}^{2m} \neq Z_1^{2m} + \cdots + Z_s^{2m}$ for any $Z_1, \ldots, Z_s \in F$.*

Proof. If $X_1^{2m} + \cdots + X_s^{2m} + 1$ is a minimizer for v (i.e. an element that attains the minimal positive value), then we set $X_{s+1} = 1$. Otherwise, we set $X_{s+1} = 1 + \pi$ for a minimizer π for v, and we see that

$$X_1^{2m} + \cdots + X_s^{2m} + X_{s+1}^{2m} = X_1^{2m} + \cdots + X_s^{2m} + 1 + \pi^2 \left(\sum_{i=2}^{2m} \binom{2m}{i} \pi^{i-2} \right) + 2m\pi$$

is also a minimizer for v. For the sake of contradiction, let us assume that $X_1^{2m} + \cdots + X_{s+1}^{2m} = Z_1^{2m} + \cdots + Z_s^{2m}$ for some $Z_1, \ldots, Z_s \in F$ with $v(Z_s) = \min\{v(Z_i) \mid 1 \le i \le s\}$. Since $Z_1^{2m} + \cdots + Z_s^{2m}$ is a minimizer for v, we have that $v(\widetilde{Z}_1^{2m} + \cdots + \widetilde{Z}_s^{2m}) > 0$, where $\widetilde{Z}_i := Z_i / Z_s$ for $1 \le i \le s-1$, since otherwise Z_s^2 would be a minimizer for v, which is impossible for a square. We obtain the contradiction that $-1 = \tilde{z}_1^{2m} + \cdots + \tilde{z}_{s-1}^{2m}$ in κ_v, where $\tilde{z}_i$ denote the residues of $\tilde{Z}_i$ in κ_v. □

Corollary 3.5. *Let m be a positive integer. Let v be a discrete valuation of rank one on a field F with residue field κ_v of characteristic relatively prime to $2m$. Then $p_{2m}(F) \ge p_{2m}(\kappa_v)$ if κ_v is real and $p_{2m}(F) \ge s_{2m}(\kappa_v) + 1$ if κ_v is nonreal.*

Applying this to Proposition 2.5, we obtain immediately the following relative lower bound for higher Pythagoras numbers of function fields.

Theorem 3.6. *Let m be a positive integer. Let F/K be a real function field of transcendence degree d. Then $p_{2m}(F) \ge p_{2m}(L(X_1, \ldots, X_{d-1}))$ for some finite real extension L/K.*

Recall from the introductory part of this section that two well known results from quadratic form theory show that $p_2(L(X_1, \ldots, X_{d-1}) \ge d - 1 + 2$ when $d - 1 \ge 2$ and L is real. Hence we deduce the following absolute lower bound:

Corollary 3.7. *Let F/K be a real function field of transcendence degree $d \geq 3$. Then $p_2(F) \geq d+1$.*

Lemma 3.8. *Let m a positive integer. Then $p_{2m}(K(X_1,\dots,X_d)) \geq d+1$ for every real field K and integer $d \geq 0$.*

Proof. Let R denote the real closure of K with respect to some ordering of K. By [12, Corollary 2], we know that there are $f_1,\dots,f_{d-1}$ such that $1+f_1^{2m}+\dots+f_{d-1}^{2m}$ is not a sum of $d-1$ powers of exponent $2m$ in $R(X_1,\dots,X_{d-1})$, and hence in particular also not in the subfield $R(f_1,\dots,f_{d-1})$. Let $\mathfrak{p}$ be the kernel of the evaluation morphism $R[X_1,\dots,X_{d-1}] \to R(f_1,\dots,f_{d-1})$ given by $X_i \mapsto f_i$ for all $1 \leq i \leq d$ By Corollary 2.2, there exists a discrete R-valuation of finite rank on $R(X_1,\dots,X_{d-1})$ whose valuation ring contains $R[X_1,\dots,X_{d-1}]$ and such that the residue homomorphism sends X_i to f_i for any $1 \leq i \leq d-1$.

By Lemma 3.3 we have that $1+X_1^{2m}+\dots+X_{d-1}^{2m}$ is not a sum of $d-1$ powers of exponent $2m$ in $R(X_1,\dots,X_{d-1})$ and thus also not in $K(X_1,\dots,X_{d-1})$. □

Remark 3.9. The simple elegant idea of considering generic sums of higher even powers in order to obtain the reduction via base change is due to K. J. Becher. It simplified my previous proof that generalized Kucharz' proof to rational function fields over arbitrary real fields, in fact more generally, to real function fields that admit a rational place.

Corollary 3.10. *Let F/K be a real function field of transcendence degree d and let m be an arbitrary positive integer m. Then $p_{2m}(F) \geq d$. Moreover, for $d \geq 3$ we have that $p_2(F) \geq d+1$.*

Proof. By Theorem 3.6 we know that $p_{2m}(F) \geq p_{2m}(L(X_1,\dots,X_{d-1}))$ for some real field L. Then Lemma 3.8 yields the statement. □

Remark 3.11. In discussions with D. Leep (not yet published), we observed that if a real field K is not hereditarily pythagorean (i.e. it admits a finite real extension with Pythagoras number 2 or higher), then every rational function field in one variable over every finite real extension of K has Pythagoras number at least 3. By Theorem 3.6, this yields that $p_2(F) \geq 3$ for every real function field in two variables F/K. In the following we show for all positive integers m that $p_{2m}(F) \geq 3$, even allowing K to be hereditarily pythagorean and F to be nonreal (of level at least 2).

Proposition 3.12. *Let m be a positive integer. Let F/K be a function field of transcendence degree $d \leq 2$ and assume that -1 is not a $2m$-th power in F. Then $p_{2m}(F) \geq d+1$.*

Proof. When $d=0$ there is nothing to show. When $d=1$, we obtain that $p_{2m}(F) \geq 2$ for the simple reason that $p_{2m}(K(X)) \geq 2$, as $1+X^{2m}$ is not a $2m$-th power in $K(X)$, and $p_{2m}(F) \neq 1$ then follows from a Going-Down result [3, Thm. 3.8] for $2m$-pythagorean finite extensions.

When $d = 2$, we invoke Corollary 2.8 that asserts the existence of a discrete valuation of rank one with nonreal residue field in which -1 is not a $2m$-th power for any positive integer m. Corollary 3.5 yields that $p_{2m}(F) \geq 3$. $\square$

Remark 3.13. The statement of Proposition 3.12 actually yields a stronger version of Theorem 1.1 in the case $d = 2$. Namely, we can replace the assumption that F is real by the weaker assumption that -1 is not a square in F (and in fact for $d = 0, 1$ even this can be omitted).

4. Perspectives and conjectures

In [17], geometrically rational $\mathbb{R}$-surfaces S without nonsingular real points were considered. They showed that -1 is a sum of two squares in its function field $\mathbb{R}(S)$, which due to the identity $X = (\frac{X+1}{2})^2 - (\frac{X-1}{2})^2$ leaves the possibility that its Pythagoras number is either 2 or 3. Together with Proposition 3.12, we obtain that $p(\mathbb{R}(S)) = 3$. Note that this observation is not a consequence of Kucharz' general lower bound $d + 1$ for real function fields over $\mathbb{R}$, since $\mathbb{R}(S)$ is nonreal in the situation of the theorem.

Clearly, if S is birational to a finite cover of a geometrically rational surface without real points, then $s(\mathbb{R}(S)) = 2$ and $p(\mathbb{R}(S)) = 3$, as well. Conversely, suppose that S is a geometrically irreducible surface without nonsingular real points and $p(\mathbb{R}(S)) = 3$, whereby -1 a sums of two squares but not a square in $\mathbb{R}(S)$. We thus obtain an embedding of the function field of the conic C defined by $X^2 + Y^2 + Z^2 = 0$ over $\mathbb{R}$ into $\mathbb{R}(S)$, and we see that S is birational to a finite covering of $C \times_{\mathbb{R}} \mathbb{A}^1$, which is a geometrically rational surface without real points.

One is thus left with the task to characterize the real $\mathbb{R}$-surfaces S such that $p(\mathbb{R}(S)) = 3$. The only real $\mathbb{R}$-surfaces for which the exact value of the Pythagoras number is known are rational surfaces for which we have $p(\mathbb{R}(S)) = 4$ (e.g. by the earlier mentioned work of Cassels–Ellison–Pfister).

Question 4.1. Is $p(\mathbb{R}(S)) = 4$ for every irreducible real $\mathbb{R}$-surface S ?

For real fields F that are finitely generated of transcendence degree d over a real closed field K, Kucharz showed in [12, Corollary 2] that $p_{2m}(F) \geq d + 1$ for any positive integer m. The question arises whether the lower bound $d + 1$ for the higher Pythagoras numbers still holds after removing the condition that the base field K is to be real closed. For $d \leq 2$ or $m = 1$ we showed this. The difficulty in our approach for the case $d \geq 3$ was that we do not know whether the lower bound $p_{2m}(F) \geq d + 2$ for rational function fields F/K still holds for $m \geq 2$. The main problem is that the Cassels–Pfister representation theorem [16, Chap.1, Thm 3.2] fails for higher even degree forms. Nevertheless, I believe that the results collected in this article justify the following conjecture.

Conjecture 4.2. *All higher Pythagoras numbers of a real field that is finitely generated of transcendence degree d over a subfield are bounded by $d + 1$ from below.*

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