A problem regarding the tracing of graphs

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A problem regarding the tracing of graphs

1. One of the first questions in elementary topology, illustrated by the *Königsberger Bridge Problem,* requires the conditions for the tracing of a plane figure in a continuous path without passing twice through any lines in the figure. EULER solved the slightly more general problem of deciding when it is possible to trace the sides in an arbitrary finite graph G continuously, passing along each side once and only once and returning to the starting point. The result, as one knows, is that the graph must be

(1) *Connected.*

(2) An *Euler graph* characterized by the property that *each vertex is of even order,* that is, it must be the joining point of an even number of sides.

The proof is simple and can be found in a considerable number of expositions so that it need not occupy us here. In the following we shall, however, discuss another problem also connected with the tracing of Euler graphs. Our starting point is the observation that when it is possible to draw a graph in one continuous trait without duplication as required, it does not follow immediately how such a tracing is obtainable. As a very simple example let us take the figure 8 shaped graph indicated in Fig. 1. When starting at the vertex a, one may first proceed in the cycle $a\,b\,c\,d\,a$ and in order to trace the whole figure it is necessary afterwards to insert the cycle $c\,e\,f\,g\,c$. This remark leads us to consider the following general problem regarding Euler graphs:

When does a connected Euler graph have the property that if one starts and returns to the same vertex a, then the whole graph is automatically traced without repetition if one proceeds according to the single rule that whenever one arrives at a vertex one shall always select some side which has not previously been traversed?

A graph with this property may be called *arbitrarily traceable from the vertex a.* One sees immediately that the graph in Fig. 1 is not arbitrarily traceable from a, but it has this property with respect to the vertex c. In Fig. 2 one has a graph which is arbitrarily traceable from a, but from no other point, while Fig. 3 is arbitrarily traceable both from a and b. Such a graph which consists of disjoint cycles intersecting only in two vertices a and b as in Fig. 3 we shall call a *skein.*

The reader readily verifies that the only graph which is arbitrarily traceable from *all* its points is a *cycle.*

2. Before we can solve the general problem of finding all arbitrarily traceable graphs it is necessary to derive certain auxiliary facts about them. Let G be a graph which is arbitrarily traceable from the vertex a; furthermore, let G_1 be some (connected)

Euler subgraph of G which also has a as one of its vertices. Then G_1 can be traced from a and after this process there remains in G_1 some uniquely determined complementary graph G_2 such that one has the *direct decomposition*

$$G = G_1 + G_2 . \tag{1}$$

But here the remaining graph G_2 must be arbitrarily traceable from a because otherwise the original graph G itself could not have this property. But when the same

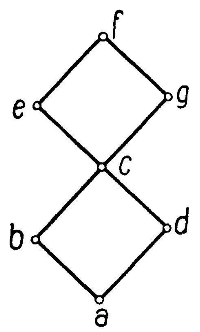

Fig. 1

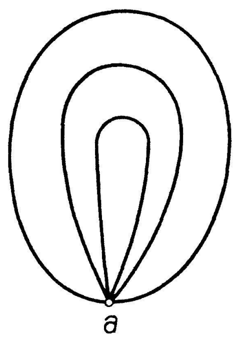

Fig. 2

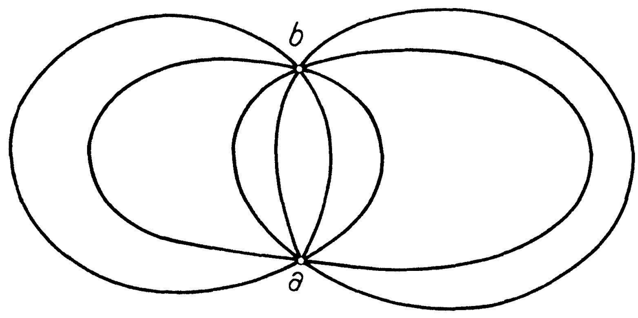

Fig. 3

argument is applied to G_2 in (1) it follows that also G_1 must be arbitrarily traceable from a and we have:

Theorem 1. Let G be a graph which is arbitrarily traceable from the vertex a. Then any (connected) Euler subgraph G_1 of G including a is also arbitrarily traceable from a and has an arbitrarily traceable complement G_2 in G.

Now let $2n$ be the order of the vertex a in the graph G. Any tracing of the graph from a must start out along some particular side S_1 and return along a different side S_2. It is therefore clear that there exists some cycle $\mathfrak{C}_1$ in G containing S_1 and S_2. Since $\mathfrak{C}_1$ is arbitrarily traceable it follows from theorem 1 that the complement $G_1 = G - \mathfrak{C}_1$ has the same property. If $\mathfrak{C}_2$ is another cycle through a in G_1 one concludes further that $G_2 = (G - \mathfrak{C}_1) - \mathfrak{C}_2$ is arbitrarily traceable and by continuing the argument one arrives at the result:

Theorem 2. A graph which is arbitrarily traceable from the vertex a of order $2n$ is the direct sum of n cycles $\mathfrak{C}_i$ through a

$$G = \mathfrak{C}_1 + \mathfrak{C}_2 + \cdots + \mathfrak{C}_n . \tag{2}$$

This represents a necessary but not a sufficient condition for an arbitrarily traceable graph. Now let us consider the case of *two* cycles $\mathfrak{C}_1$ and $\mathfrak{C}_2$ contained in such a graph G. We suppose that $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are disjoint, that is, have no side in common, and that they intersect in a and possibly in a certain number of other vertices of G. The graph $\mathfrak{C}_1 + \mathfrak{C}_2$ is then an Euler graph and therefore arbitrarily traceable according to theorem 1.

We now select a definite direction on $\mathfrak{C}_1$ and proceed from a to the first vertex b which $\mathfrak{C}_1$ has in common with $\mathfrak{C}_2$ and similarly, in the opposite direction on $\mathfrak{C}_1$ let c be the first common vertex (see Fig. 4). To trace the graph $\mathfrak{C}_1 + \mathfrak{C}_2$ let us begin at a and proceed in the given direction to b and return along that part of $\mathfrak{C}_2$ which does not include c. Then we proceed from a to c along $\mathfrak{C}_1$ in the opposite direction and return to a along that part of $\mathfrak{C}_2$ which does not contain b. After this process we have no further exit from a so that the complete graph $\mathfrak{C}_1 + \mathfrak{C}_2$ must have been traced. But it is clear that if $b \neq c$ the section of $\mathfrak{C}_1$ between b and c not including a has not been covered so that we conclude $b = c$. This gives the result:

Theorem 3. Let G be a graph arbitrarily traceable from the vertex a and $\mathfrak{C}_1$ and $\mathfrak{C}_2$ two cycles in G without common sides and passing through a. Then $\mathfrak{C}_1$ and $\mathfrak{C}_2$ can intersect in at most one other vertex.

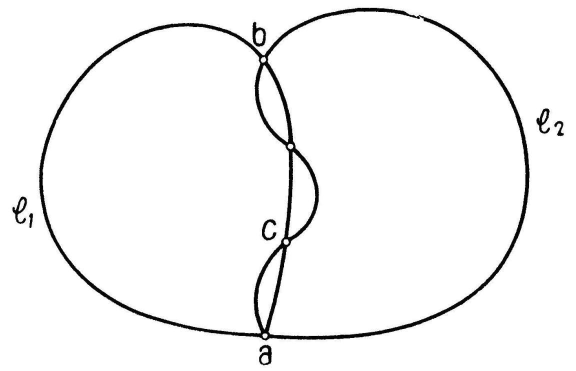

Fig. 4

3. We are now ready to deduce the following criterion:

Theorem 4. The necessary and sufficient condition that an Euler graph be arbitrarily traceable from a vertex a is that it contain no cycles not including the vertex a.

Proof. Let us suppose first that there exists some cycle $\mathfrak{C}$ in G which does not pass through a. We form the graph $G_1 = G - \mathfrak{C}$. Since G and $\mathfrak{C}$ are Euler graphs all vertices in G_1 must be of even order so that G_1 is also an Euler graph. G_1 need not be connected but it has a maximal connected component $G^{(a)}$ including the vertex a. Then the sum

$$G_2 = G^{(a)} + \mathfrak{C}$$

must be a connected Euler graph and both G_2 and $G^{(a)}$ must be arbitrarily traceable according to theorem 1. But this leads to a contradiction since one can begin tracing G_2 by first tracing $G^{(a)}$ from a and when one has returned to a there is no further exit to reach the cycle $\mathfrak{C}$.

On the other hand, if the Euler graph G contains no cycle not passing through a it is connected and traceable. If it were not arbitrarily traceable one could trace a part G_1 of G exhausting all exit possibilities from a. But then the complementary graph G_2 to G_1 would also be an Euler graph, and since it is clear that an Euler graph always contains a cycle we would have a cycle in G not including a, contrary to assumption.

Theorem 4 gives a simple criterion for the graphs which are arbitrarily traceable from more than one vertex. With our previous definition we have

Theorem 5. A graph which is arbitrarily traceable from two vertices is a skein.

Proof: Let a and b be the two vertices from which the graph G is arbitrarily traceable. According to theorem 4 every cycle in G goes through both a and b. In the representation (2) of G as the direct sum of cycles every cycle $\mathfrak{C}_i$ goes through a and b and theorem 3 shows that none of them can have any other points in common. This proves theorem 5.

4. Theorem 4 makes it possible to give a simple construction of all graphs which are arbitrarily traceable. Let us denote by S_a the *star* or subgraph of G which consists of all those sides of G which are joined at the vertex a; furthermore, let G_1 be the complement of S_a in G_1 hence (see Fig. 5)

$$G = S_a + G_1.$$

Then according to theorem 4 the necessary and sufficient condition that G shall be

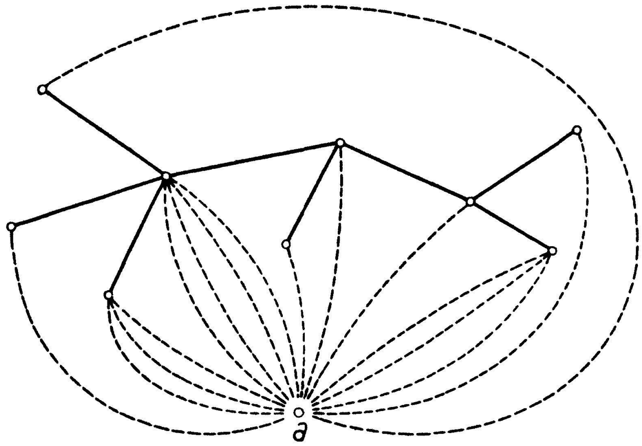

Fig. 5

arbitrarily traceable is that the graph G_1 contain no cycles, that is, it must be a topological tree.

Now let us proceed in the opposite direction and assume that some tree G_1 is given. We select a new vertex a and draw sides from a to the vertices of G_1 such that the vertices of G_1 in the new graph G have an even order. This may be achieved by drawing a single side from a to the odd vertices in G_1 and none to the even ones, but one can also, more general, draw an odd number of sides in the first case and an arbitrary even number of sides in the second. To show that the resulting graph is an Euler graph it is only necessary to verify that the order of the vertex a is even, since the other vertices are already even. But this is an immediate consequence of the relation

$$2\,s = \Sigma\,\mu_i$$

for any graph, where s is the number of sides and μ_i the order of the i-th vertex. Thus we have:

Theorem 6. One can construct all graphs which are arbitrarily traceable by taking a topological tree T and join each vertex of T by a number of sides to a new vertex a in such a manner that in the resulting graph each vertex is of even order.

The construction is illustrated in Fig. 5.

Since a tree is a planar graph it is clear that all arbitrarily traceable graphs must be planar.

5. The problem which we have discussed in the preceding may be considered as a problem of constructing a set of roads such that when one always follows new paths at each intersection all paths will be covered a single time and one returns to the starting point. Such a pattern would be suitable for the lay-out of an exposition.

There are several similar questions which one may discuss. If one supposes that the roads are lined with shops and one will cover all roads once in both directions, this is always possible, as one easily realizes. But one may restrict the paths by requiring that one shall not be permitted to return along the same road immediately from any of the intersections; then certain restrictions must be imposed on the graph. One may also ask when it is possible to cover the graph in this manner by *any* route if one only follows the rule that a new path shall be selected whenever one reaches an intersection. I leave some of these problems to the study of the reader.

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Ein zeichnerisches Lösungsverfahren für Differentialgleichungen zweiter Ordnung

Ist eine Differentialgleichung zweiter Ordnung in der explizit darstellbaren Form gegeben

$$y'' = f(x, y, y'),$$

so läßt sich zu ihrer genäherten zeichnerischen Integration ein Verfahren verwenden, das nicht nur zu einer ersten raschen Orientierung über den Lösungsverlauf geeignet ist (wenn von Singularitäten abgesehen wird), sondern das auch so genau ausgeführt werden kann, daß es ohne weiteres den üblichen praktischen Erfordernissen genügen dürfte. Das Verfahren stützt sich lediglich auf elementare Eigenschaften der gewöhnlichen quadratischen Parabel, aus der man sich die gesuchte Integralkurve stückweise zusammengesetzt denkt. Ein einzelnes solches Stück ist in der Fig. 1 dargestellt.

Für eine quadratische Parabel gilt bekanntlich, daß die Abszisse x des Schnittpunktes T zweier benachbarter Tangenten I und II gerade in der Mitte zwischen den Abszissen ihrer beiden Berührungspunkte *1* und *2* liegt. Haben diese den Abstand Δx voneinander, so läßt sich wegen des geradlinigen Verlaufes der ersten Ableitung y' die zweite Ableitung y'' im Parabelpunkte P mit der Abszisse x wie folgt ausdrücken:

$$y'' = \frac{y_2' - y_1'}{\Delta x} = \frac{1}{\Delta x}\left(\frac{\Delta y_2}{\Delta x} - \frac{\Delta y_1}{\Delta x}\right),$$

oder mit der zeichnerisch bequemeren Benutzung ähnlich vergrößerter Dreiecke:

$$y'' = \frac{1}{\Delta x}\left(\frac{k_2}{h} - \frac{k_1}{h}\right) = \frac{\Delta k}{h\,\Delta x}.$$

Hat man also beispielsweise die konstante zweite Ableitung y'' einer quadratischen Parabel gegeben, so lassen sich nach der Wahl einer Anfangstangente I in einem Anfangspunkt *1* weitere Tangenten II, ... auf folgende Weise zeichnerisch finden: