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IX lautet dann:

X $\left\{\begin{array}{l} -h'u - u'Gu \quad (24') \\ \text{ist zu maximieren unter den Nebenbedingungen} \\ u \geq 0. \quad (26) \end{array}\right.$

Dabei ist zur Abkürzung $h = AC^{-1}p + b$ und $G = \frac{1}{2}AC^{-1}A'$ gesetzt. $\boldsymbol{x}$ stellt dann und nur dann eine Lösung von VIII dar, wenn $\boldsymbol{x} = x(\boldsymbol{u})$, wobei $\boldsymbol{u}$ eine Lösung von X darstellt.

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LITERATURVERZEICHNIS

Krelle, W., Kuenzi, H. P., *Lineare Programmierung* (Industrielle Organisation, Zürich 1958).

Kuenzi, H. P., Krelle, W., *Nichtlineare Programmierung*. (Springer-Verlag, Berlin–Göttingen–Heidelberg, 1962).

Vajda, S., *Mathematical Programming* (Addison-Wesley, London 1961).

Zoutendijk, G., *Methods of feasible Directions* (Elsevier, Amsterdam 1960).

Arithmetical Notes, XIII. A Sequel to Note IV*)

1. *Introduction*

First we introduce some terminology and notation. For any positive integer n,

$$n = p_1^{e_1} \cdots p_r^{e_r}, \tag{1.1}$$

where $p_1, \ldots, p_r$ are the prime divisors of n, let $t(n)$ denote the maximal multiplicity of n, that is, $t(n) = \max(e_1, \ldots, e_r)$, $t(1) = 0$. For arbitrary positive integers k, let n_k denote the k-segment of n, meaning the number $n_k = n/m$ where m is the largest k-th power divisor of n. Further, let $t_k(n)$ denote the maximal multiplicity of the k-segment of n, $t_k(n) = t(n_k)$.

In this note we prove elementary estimates for the distribution of two sequences of integers, $U_{k,t}$ and $V_{k,t}$, defined as follows. Let t denote a fixed positive integer; define $U_{k,t}$ to be the set of all n for which $t_k(n) < t$ and $V_{k,t}$ to be the set of all n for which $t_k(n) = t$. The result obtained for $U_{k,t}$ is given in Theorem 1 and that for $V_{k,t}$ in Theorem 2. It will be noted that $U_{k,k}$ is the set of all positive integers while $U_{k,1}$ is the set of k-th powers. More generally, $U_{k,t}$ is the set of all n whose k-segment is a t-free integer (an integer whose largest t-th power divisor is 1). Note moreover that the sets $V_{k,t}$ contain no k-th power integers.

We mention here a single special case of Theorem 1. Let $H_k(x)$ denote the number of $n \leq x$ contained in $H_k = U_{k,2}$, that is, $H_k(x)$ is the number of those n, not exceeding x, whose k-segment is square-free. Then for $k > 2$,

$$H_k(x) = \left(\frac{\zeta(k)}{\zeta(2)}\right) x + O\left(\sqrt{x}\right), \tag{1.2}$$

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where $\zeta(k)$ is the sum of the series, $\sum_{n=1}^{\infty} n^{-k}$. This result is equivalent to one contained in [1, Corollary 2.1]. For a slight generalization we mention [3, Lemma 2.2].

The results of this paper extend those of an earlier paper [2]. In fact, the main results of [2] are deduced as corollaries in §§ 2 and 3.

2. *The sequences* $U_{k,t}$

The enumerative function $G(x)$ of a set of integers G is the number of $n \leqq x$ contained in G; in particular, $H_k(x)$ in (1.2) is the enumerative function of H_k. The characteristic function $g(n)$ of G is the function defined to be 1 for all $n \in G$, 0 for all $n \notin G$. In accord with this notation, $U_{k,t}(x)$ will be used to represent the enumerative function of $U_{k,t}$ and $u_{k,t}(n)$ the characteristic function.

We recall first some well known facts which will be used in the proof of Theorem 1. Let $q_k(n)$ denote the characteristic function of the k-free integers and $\mu(n)$ the Möbius function. Then

$$q_t(n) = \sum_{d^t e = n} \mu(d)\,; \qquad \frac{1}{\zeta(k)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k}\,, \qquad k > 1\,, \tag{2.1}$$

also we have the simple estimates,

$$\sum_{n > x} \frac{1}{n^s} = O\left(\frac{1}{x^{s-1}}\right) \quad \text{if} \quad s > 1\,, \tag{2.2}$$

$$\sum_{n \leqq x} n^s = O(x^{s+1}) \quad \text{for} \quad s > -1\,. \tag{2.3}$$

Lemma 1. *If* $k \geqq t$, *then*

$$u_{k,t}(n) = \sum_{d^k \Delta = n} q_t(\Delta) = \sum_{d^k \delta^t e = n} \mu(\delta)\,. \tag{2.4}$$

Proof. Every n has a unique representation as a product of a k-th power and a k-free integer. Hence, since $t \leqq k$, the only term in the first sum of (2.4) which may be different from 0 is the one corresponding to the factorization, $n = d^k \Delta$, Δ k-free, so that the sum reduces to $q_t(\Delta) = u_{k,t}(n)$. Application of $(2.1)_1$ yields the lemma.

Let $[x]$ denote the largest integer $\leqq x$.

Theorem 1. *If* $1 < t < k$, *then for* $x \geqq 1$

$$U_{k,t}(x) = \left(\frac{\zeta(k)}{\zeta(t)}\right) x + O\left(\sqrt[t]{x}\right). \tag{2.5}$$

Remark 1. The relation (2.5) holds trivially in case $k = t$.

Proof. By Lemma 1,

$$U_{k,t}(x) = \sum_{n \leqq x} u_{k,t}(n) = \sum_{d^k \delta^t e \leqq x} \mu(\delta) = \sum_{\delta^t \leqq x} \mu(\delta) \sum_{d^k e \leqq \frac{x}{\delta^t}} 1 = \sum_{\delta \leqq x^{1/t}} \mu(\delta) \sum_{d^k \leqq \frac{x}{\delta^t}} [x/d^k \delta^t]$$

$$= \sum_{\delta \leqq x^{1/t}} \mu(\delta) \sum_{d \leqq \left(\frac{x}{\delta^t}\right)^{1/k}} \left(\frac{x}{d^k \delta^t} + O(1)\right) = \sum_{\delta \leqq x^{1/t}} \mu(\delta) \left\{\frac{x}{\delta^t} \sum_{d \leqq \left(\frac{x}{\delta^t}\right)^{1/k}} d^{-k} + O\left(\left(\frac{x}{\delta^t}\right)^{1/k}\right)\right\}.$$

Hence by (2.2), since $k > 1$,

$$U_{k,t}(x) = \sum_{\delta \leqq x^{1/t}} \mu(\delta) \left\{ \frac{x\zeta(k)}{\delta^t} - \frac{x}{\delta^t} \sum_{d > \left(\frac{x}{\delta^t}\right)^{1/k}} d^{-k} + O\left(\left(\frac{x}{\delta^t}\right)^{1/k}\right) \right\}$$

$$= x\,\zeta(k) \sum_{\delta \leqq x^{1/t}} \mu(\delta)\,\delta^{-t} + O\left(x^{1/k} \sum_{\delta \leqq x^{1/t}} \delta^{-t/k}\right),$$

and by $(2.1)_2$, the boundedness of $\mu(n)$, and the hypothesis, $t > 1$,

$$U_{k,t}(x) = \frac{x\zeta(k)}{\zeta(t)} + O\left(x \sum_{\delta > x^{1/t}} \delta^{-t}\right) + O\left(x^{1/k} \sum_{\delta \leqq x^{1/t}} \delta^{-t/k}\right).$$

By (2.2) and (2.3), the O-terms are $O(x^{1/t})$, and the theorem is proved.

Let U_k denote the set of all n in (1.1) such that $e_i \not\equiv -1 \pmod{k}$, $i = 1, \ldots, r$, and let $U_k(x)$ be the enumerative function of U_k. It is not difficult to see that $U_k = U_{k,k-1}$ if $k > 1$. Hence the theorem becomes, in the extreme case $t = k - 1$,

Corollary 1 (cf. [2, Theorem 3.1]). *If* $k > 2$,

$$U_k(x) = \left(\frac{\zeta(k)}{\zeta(k-1)}\right) x + O\left(x^{\frac{1}{k-1}}\right). \tag{2.6}$$

The other extreme case of the theorem, $t = 2$, yields the formula stated in the Introduction.

Remark 2. In the case $t = k - 1$, the first half of Lemma 1 is equivalent to the identity proved in [2, Lemma 2.2]. The proof given in [2] is based on an inversion argument which is somewhat longer, though perhaps more instructive, than the one of the present paper.

3. *The sequences* $V_{k,t}$

Let $V_{k,t}(x)$ be the enumerative function of $V_{k,t}$. Now

$$V_{k,t}(x) = U_{k,t+1}(x) - U_{k,t}(x), \quad 1 \leqq t < k. \tag{3.1}$$

Noting, in addition, that $U_{k,k}(x) = [x]$, $U_{k,1}(x) = \left[\sqrt[k]{x}\right]$, we have from Theorem 1

Theorem 2. *If* $1 < t < k$, *then*

$$V_{k,t}(x) = c_{k,t}\,x + O\left(\sqrt[t]{x}\right), \tag{3.2}$$

where

$$c_{k,t} = \begin{cases} \zeta(k)\,(\zeta^{-1}(t+1) - \zeta^{-1}(t)) & \textit{if} \quad t < k-1, \\ 1 - \dfrac{\zeta(k)}{\zeta(k-1)} & \textit{if} \quad t = k-1; \end{cases} \tag{3.3}$$

if $k > 2$,

$$V_{k,1}(x) = \left(\frac{\zeta(k)}{\zeta(2)}\right) x + O\left(\sqrt{x}\right). \tag{3.4}$$

Let $\tau(n)$ denote the number of divisors of n and let V_k represent the sequence of n for which $k \mid \tau(n)$. As usual, $V_k(x)$ is the enumerative function of V_k. It is easily deduced that if k is an odd prime, then $V_k = V_{k,k-1}$, $V_k(x) = V_{k,k-1}(x)$. Hence the case $t = k - 1$ of the theorem leads to

Corollary 1 ([2, (1.3)]). *If k is a prime $\neq$ 2, then*

$$V_k(x) = \left(1 - \frac{\zeta(k)}{\zeta(k-1)}\right) x + O\left(x^{\frac{1}{k-1}}\right). \tag{3.5}$$

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BIBLIOGRAPHY

[1] ECKFORD COHEN, *Arithmetical Notes*, II. *An estimate of* ERDÖS *and* SZEKERES, Scripta Mathematica, to appear.

[2] ECKFORD COHEN, *Arithmetical Notes*, IV. *A set of integers related to the divisor function*, Journal of the Tennessee Academy of Science, vol. *37* (1962), 119–120.

[3] ECKFORD COHEN, *Arithmetical Notes*, V. *A divisibility property of the divisor function*, American Journal of Mathematics, vol. *83* (1961), 693–697.

Kleine Mitteilungen

Adjungierte Sekanten und Tangenten zweier Kreise

Man betrachte irgendzwei in einer Ebene liegende und sich im Punkte S schneidende Kreise K_1 und K_2; der Schnittwinkel sei α. Durch S lege man zwei beliebige Geraden p und q, die mit K_1 bzw. K_2 ausser S im allgemeinen je einen Schnittpunkt P_1, Q_1 bzw. P_2, Q_2 ergeben. Die durch P_1, Q_1 bzw. P_2, Q_2 verlaufenden Geraden g_1 bzw. g_2 wollen wir ein Paar «adjungierte Sekanten» nennen. Im Fall $p = q$ betrachten wir die durch $P_1 = Q_1$ bzw. $P_2 = Q_2$ verlaufenden «adjungierten Tangenten».

Untersucht man nun die Menge der Schnittpunkte je zweier zueinander adjungierter Sekanten und Tangenten bei willkürlicher Variation von p und q, so ergeben sich einige interessante Eigenschaften.

Man kann zu dieser Fragestellung ausgehend von verschiedenen geometrischen Aspekten mit den entsprechenden Vorkenntnissen Zugang finden. Bemerkenswert ist aber auch die Möglichkeit einer elementaren Betrachtungsweise, die im folgenden dargelegt werden soll.

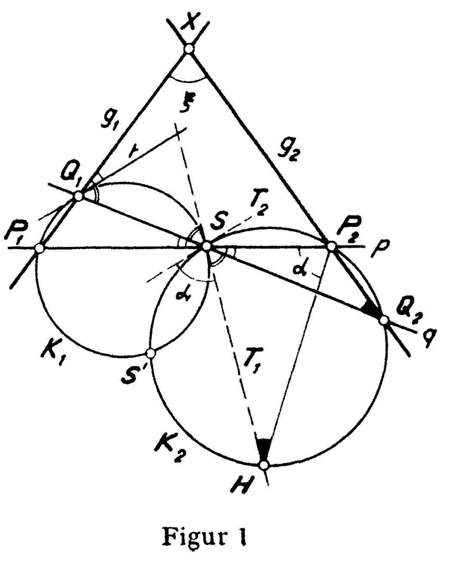
Wir vereinbaren vorerst die Bezeichnungen: S, S' Schnittpunkt der Kreise K_1, K_2 mit S als Sekantenzentrum. Die interessierenden Schnittpunkte adjungierter Sekanten (bzw. Tangenten) g_1, g_2 nennen wir «X-Punkte».

1. *Satz I.* Adjungierte Sekanten (und Tangenten) schneiden sich stets unter demselben Winkel, und zwar unter dem Schnittwinkel α der Kreise.

Den Beweis hierfür entnehme man in einfacher Weise aus der folgenden Skizze bei Beachtung elementargeometrischer Tatsachen.

Dabei bedeuten:

t die Tangente an K_1 in Q_1
T_1 die Tangente an K_1 in S
T_2 die Tangente an K_2 in S



Figur 1