Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 18 (1963)
Heft: 3

Artikel: On the diameter and triameter of a convex body
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DOI: https://doi.org/10.5169/seals-22637

Posons maintenant

$$u_n = 3\,b_n - a_n, \quad v_n = 4\,b_n, \quad w_n = 3\,b_n + a_n \quad \text{pour} \quad n = 1, 2, \ldots. \tag{4}$$

D'après (1) et (2) on a $u_n < v_n < w_n$ pour $n = 1, 2, \ldots$ et $u_1 = 1$, $v_1 = 4$, $w_1 = 5$, $u_{n+1} = 3\,b_{n+1} - a_{n+1} = 35\,a_n + 71\,b_n$ pour $n = 1, 2, \ldots$, donc les nombres u_n, v_n, w_n sont, pour $n = 2, 3, \ldots$, des entiers > 1 et, les nombres b_n croissant avec n, on a $v_{n+1} > v_n$ pour $n = 1, 2, \ldots$

Or, d'après (4), on trouve pour $n = 1, 2, \ldots$

$$u_n^3 + w_n^3 - 2\,v_n^3 - u_n - w_n + 2\,v_n = 2\,b_n\,[(3\,a_n)^2 - 37\,b_n^2 + 1],$$

donc, d'après (3):

$$u_n^3 + w_n^3 - 2\,v_n^3 - u_n - w_n + 2\,v_n = 0 \quad \text{pour} \quad n = 1, 2, \ldots,$$

d'où

$$\frac{u_n^3 - u_n}{6} + \frac{w_n^3 - w_n}{6} = 2\,\frac{v_n^3 - v_n}{6} \quad \text{pour} \quad n = 1, 2, \ldots,$$

donc

$$T_{u_n-1} + T_{w_n-1} = 2\,T_{v_n-1} \quad \text{pour} \quad n = 1, 2, \ldots, \tag{5}$$

ce qui prouve que, pour $n = 1, 2, \ldots$, les trois nombres tétraédraux

$$T_{u_n-1}, \quad T_{v_n-1} \text{ et } T_{w_n-1}$$

forment une progression arithmétique. Comme $u_n < v_n < w_n$ et $v_{n+1} > v_n$ pour $n = 1, 2, \ldots$, il en résulte notre théorème. Pour $n = 2$ on obtient $T_{140} + T_{728} = 2\,T_{579}$.

Pour $n = 1$ on a $u_1 = 1$, $v_1 = 4$, $w_1 = 5$ et la formule (5) donne $T_4 = 2\,T_3$. Dans ma note citée j'ai posé le problème s'il existe d'autres solutions en nombres naturels m et n de l'équation $T_m = 2\,T_n$. M. S. L. SEGAL a démontré récemment qu'il n'existe pas d'autres solutions[2]). Or, il mentionne aussi (l.c., p. 638) que M. S. CHOWLA a démontré récemment qu'il existe une infinité de nombres tétraédraux qui sont sommes de deux nombres tétraédraux (ce que j'ai démontré dans ma note citée des Elemente der Math.). La démonstration de M. CHOWLA m'est inconnue.

Il est encore à remarquer qu'il existent d'autres solutions de l'équation $T_x + T_y = 2\,T_z$ outre celles que nous avons trouvées, par exemple $T_4 + T_{10} = 2\,T_8$.

Or, M. A. MĄKOWSKI a posé le problème suivant, dont la solution me semble être difficile: *Existe-t-il pour tout nombre naturel k une infinité de solutions de l'équation $T_x + T_y = k\,T_z$ en entiers positifs x, y* et *z?*

Je sais démontrer (ce que je ferai ailleurs) qu'il existe une infinité de nombres naturels k pour lesquels cela est vrai. W. SIERPIŃSKI (Varsovie)

On the Diameter and Triameter of a Convex Body

1. By a *convex body* in Euclidean n-dimensional space E_n we shall mean a compact, convex subset with interior points. One phase of the theory of convex bodies seeks to establish inequalities between the geometrical invariants associated with these bodies.

[2]) S. L. SEGAL, *A note on pyramidal numbers*, Amer. Math. Monthly *69* (1962), p. 637.

The most famous of such inequalities, for example, is the so-called *isoperimetric inequality: Let the convex body K have volume V and surface area A. Then*

$$A^n \geq n^n \pi_n V^{n-1}, \tag{1}$$

where equality can hold if and only if K is spherical. The constant π_n is the volume of the unit ball in E_n. For a proof of (1) see [2][1]), p. 109.

The inequality (1) is quite powerful and requires some effort to prove. In this note we wish to show how an interesting technique from the theory of convex bodies can be used to establish similar, though weaker, inequalities. The technique to which we refer relies on the following formula, due to CAUCHY: *Let the convex body K have surface area A. Given a point ω on the unit sphere centered at the origin, let E_ω be the hyperplane through the origin orthogonal to the segment joining the origin to ω. The orthogonal projection, K_ω, of K onto E_ω is a convex body with respect to E_ω. Let V_ω be the (n-1-dimensional) volume of K with respect to E_ω. Then*

$$A = \frac{1}{\pi_{n-1}} \int V_\omega \, d\omega , \tag{2}$$

where the integration is over the entire surface of the unit sphere. A proof of this remarkable relation is given in [2], p. 48.

It is evident that mathematical induction, used in conjunction with (2), might be useful in establishing geometric inequalities involving surface area. Indeed, in §2 we use this technique in deriving a well known inequality of BIEBERBACH (see formula (3)). The §3 is devoted to proving a similar inequality involving the volume and «triameter» of a convex body (see formula (9)). Our proofs are simple, but they suffer from the defect of relying on (1) or equally difficult inequalities. This is because formula (2) enables us to apply induction simply where surface area is involved, but no such useful formula exists for the volume.

2. By the ***diameter*** of the convex body K we shall mean the length of the longest segment contained in K. We then have the *inequality of* BIEBERBACH: *Let K have diameter D and volume V.* Then

$$V \leq \pi_n \left(\frac{D}{2}\right)^n, \tag{3}$$

with equality holding if and only if K is spherical. As usual, π_n denotes the volume of the unit ball in E_n. The theorem can be restated in the form: *Among all convex bodies of the same diameter, the sphere has the largest volume.* Proofs are given in [2], p. 76 and p. 107. A direct geometrical proof without relying on (1) is given in [3], p. 173. Our proof is as follows.

Proof. Let D_ω be the diameter of the orthogonal projection K_ω of K onto hyperplane E_ω. Then evidently $D_\omega \leq D$. Then if the theorem is true in n-1-dimensional space, we have

$$V_\omega \leq \pi_{n-1} \left(\frac{D_\omega}{2}\right)^{n-1} \leq \pi_{n-1} \left(\frac{D}{2}\right)^{n-1} \tag{4}$$

Applying (2), we have, by (4),

$$A \leq n \, \pi_n \left(\frac{D}{2}\right)^{n-1}. \tag{5}$$

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[1]) Numbers in brackets refer to References, page 57.

Thus, by (1) and (5),

$$n^n \pi_n V^{n-1} \leq A^n \leq n^n \pi_n^n \left(\frac{D}{2}\right)^{n(n-1)}, \tag{6}$$

and (3) follows. If equality holds in (3), then, by (6), it must hold in (1), and the uniqueness clause of the isoperimetric theorem yields that K is spherical. To complete the induction, we establish the theorem for $n = 2$, but this is just a repetition of steps (4), (5), and (6), with $n = 2$, equality holding in the first inequality in (4).

3. Let S be the (2-dimensional) area of the largest (in the sense of area) triangle contained in the convex body K. Then by the *triameter* of K we shall mean

$$T = \left(\frac{4\,S}{3\sqrt{3}}\right)^{1/2}. \tag{7}$$

The following inequality is due to BLASCHKE [1], p. 49: *Let K be a convex body in E_2 with triameter T and (2-dimensional) volume V. Then*

$$V \leq \pi\, T^2, \tag{8}$$

with equality holding if and only if K is an ellipse. An elegant proof of a generalization of this theorem to convex n-gons is given in [4], p. 36. We can generalize the theorem in another direction as follows: *Let K be a convex body in E_n with triameter T and volume V. Then*

$$V \leq \pi_n\, T^n, \tag{9}$$

and equality holds if and only if 1), $n = 2$ *and K is an ellipse, or* 2), $n > 2$ *and K is spherical.*

Proof. Let $n > 2$ and let T_ω be the triameter of the orthogonal projection K_ω of K onto hyperplane E_ω. Then it is immediate that $T_\omega \leq T$. Assuming the theorem for $n - 1$, we have

$$V_\omega \leq \pi_{n-1}\, T_\omega^{n-1} \leq \pi_{n-1}\, T^{n-1}. \tag{10}$$

By (2) and (10),

$$A \leq n\, \pi_n\, T^{n-1}. \tag{11}$$

Then by (1) and (11),

$$n^n \pi_n V^{n-1} \leq A^n \leq n^n \pi_n^n\, T^{n(n-1)}. \tag{12}$$

If equality holds in (9), then it must hold in (1), because of (12), so that K is spherical. The induction is completed by using (8).

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