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den, aber ε nächstliegenden Gitterpunkt des Zylinders Z . Unter den Hyperebenen (16) einer Verteilung (2) gibt es also im vorliegenden Fall keine ε beliebig benachbarte Hyperebene. Entsprechendes folgt für jede Hyperebene der Verteilung (2). Somit gilt:

Sind die Verhältnisse der Logarithmen von $|x_1|, \dots, |x_n|$ rational, so besitzt die Verteilung (2) (und damit auch jede ihrer unendlichen Teilverteilungen) keinen eigentlichen Häufungspunkt.

b) Die Gleichung (17) von ε hat *mindestens zwei Koeffizienten mit irrationalem Verhältnis*. Es sei etwa $\lg |x_1|/\lg |x_2|$ irrational. Die Schnittgerade der Hyperebene ε mit der Koordinatenebene $i_1 i_2$ hat somit die Gleichung

$$i_2 = - \frac{\lg |x_1|}{\lg |x_2|} i_1. \tag{18}$$

Da die Irrationalzahl $\lg |x_1|/\lg |x_2|$ beliebig genau rational approximiert werden kann, liegt in der $i_1 i_2$ -Ebene in jedem um die Gerade (18) abgegrenzten Parallelstreifen mindestens ein Gitterpunkt. In diesem Fall gibt es also in beliebiger Nähe der Hyperebene ε noch weitere Scharebenen, und Entsprechendes folgt für jede Hyperebene der Verteilung (2). Somit gilt:

Ist mindestens ein Verhältnis der Logarithmen von $|x_1|, \dots, |x_n|$ irrational, so ist jeder eigentliche Punkt der Verteilung (2) Häufungspunkt. O. GIERING, Stuttgart

LITERATURVERZEICHNIS

- [1] H. PRÜFER: *Projektive Geometrie* (Geest und Portig, Leipzig 1953).
- [2] K. FLADT: *Analytische Geometrie spezieller ebener Kurven* (Akademische Verlagsgesellschaft, Frankfurt a.M. 1962).
- [3] J. ACZEL: *Vorlesungen über Funktionalgleichungen und ihre Anwendungen* (Birkhäuser, Basel und Stuttgart 1961).
- [4] H. MINKOWSKI: *Geometrie der Zahlen* (B. G. Teubner, Leipzig und Berlin 1910).
- [5] N. H. KUIPER: *Wiskundige Opgaven*, Deel 20, No. 3, Opgaven 111, 112 (P. Noordhoff, Groningen 1957).

Inequalities Concerning the Inradius and Circumradius of a Triangle

The purpose of this paper is to establish in an elementary manner a number of new inequalities between the inradius and circumradius of a triangle, its area and certain functions of its sides.

These inequalities will be strong enough to permit us to deduce without difficulty several well-known results, and to sharpen some inequalities due to other authors.

1. Some Identities

If T be any triangle and A_1, A_2, A_3 its vertices, we shall denote by a_i the side opposite vertex A_i , and by α_i the angle at A_i . Further, let r be the inradius and R the circumradius of T , while Δ denotes its area and $2s$ its perimeter. For simplicity, we shall write

$$\sum a_i \text{ for } \sum_{i=1}^3 a_i, \quad \sum_{i < j} a_i a_j \text{ for } \sum_{1 < i < j < 3} a_i a_j, \quad \text{and } \prod a_i \text{ for } \prod_{i=1}^3 a_i.$$

We begin by establishing several useful relations between r , R and the sides of T . It can be shown without difficulty that in any triangle,

$$2 \sum_{i < j} a_i a_j - \sum a_i^2 = 4 r (4 R + r). \quad (1)$$

For $r = \Delta/s$ and $4 r R = a_1 a_2 a_3/s$, so that

$$4 r (4 R + r) = \frac{4 a_1 a_2 a_3}{s} + \frac{4 \Delta^2}{s^2}.$$

Since $\Delta^2 = s (s - a_1) (s - a_2) (s - a_3)$, we have

$$\begin{aligned} 4 r (4 R + r) &= \frac{1}{2 s} [8 a_1 a_2 a_3 + (a_1 + a_2 - a_3) (a_2 + a_3 - a_1) (a_3 + a_1 - a_2)] \\ &= \frac{1}{2 s} [2 (a_1 + a_2 + a_3) (a_1 a_2 + a_2 a_3 + a_3 a_1) \\ &\quad - (a_1 + a_2 + a_3) (a_1^2 + a_2^2 + a_3^2)], \end{aligned}$$

whence identity (1).

Combining this relation with the algebraic identity

$$2 \sum_{i < j} a_i a_j + \sum a_i^2 = (\sum a_i)^2, \quad (2)$$

we obtain the relations

$$(\sum a_i)^2 = 2 \sum a_i^2 + 4 r (4 R + r), \quad (3)$$

and

$$4 \sum_{i < j} a_i a_j = (\sum a_i)^2 + 4 r (4 R + r). \quad (4)$$

2. Three Inequalities

Consider triangle HIO , where H , I , O denote the orthocenter, incenter, and circumcenter of T . HOBSON [1]¹⁾ proved the following formula for the distance between H and I (in our notation):

$$IH^2 = 2 r^2 - 4 R^2 \Pi \cos \alpha_i. \quad (5)$$

But

$$4 R^2 \Pi \cos \alpha_i = \frac{1}{4} (\sum a_i)^2 - (2 R + r)^2$$

(see [2] for a proof), and thus

$$IH^2 = 3 r^2 + 4 r R + 4 R^2 - \frac{1}{4} (\sum a_i)^2. \quad (6)$$

This establishes the inequality

$$(\sum a_i)^2 \leq 4 (3 r^2 + 4 r R + 4 R^2), \quad (7)$$

with equality only in an equilateral triangle, for only then do points I and H coincide.

¹⁾ Numbers in brackets refer to references, page 131.

Now let G be the centroid of T . According to a well-known theorem of EULER, G lies on OH , between O and H , and $GO:GH = 1:2$. Consequently, we have

$$GI^2 = \frac{2}{3} IO^2 + \frac{1}{3} IH^2 + \frac{2}{9} OH^2. \tag{8}$$

It is readily shown that

$$\sum a_i^2 = 3 \sum GA_i^2, \tag{9}$$

and that for any point P in the plane of T ,

$$\sum PA_i^2 = \sum GA_i^2 + 3 PG^2. \tag{10}$$

If we choose for P the circumcenter O of T , we have, by (9):

$$3 R^2 = \frac{1}{3} \sum a_i^2 + 3 OG^2,$$

whence, since $OH = 3 OG$:

$$OH^2 = 9 R^2 - \sum a_i^2. \tag{11}$$

Finally, by EULER's formula for the distance separating a triangle's incenter and circumcenter, we have

$$IO^2 = R^2 - 2 r R. \tag{12}$$

By substituting (6), (11) and (12) in formula (8), we obtain

$$GI^2 = r^2 - \frac{1}{12} (\sum a_i)^2 + \frac{2}{9} \sum a_i^2,$$

which, by relation (3), is equivalent to

$$36 GI^2 = (\sum a_i)^2 + 20 r^2 - 64 r R, \tag{13}$$

whence

$$(\sum a_i)^2 \geq 4 r (16 R - 5 r), \tag{14}$$

with equality only when the triangle is equilateral.

We have thus proved the following inequalities for a triangle with sides a_1, a_2, a_3 , inradius r , circumradius R and area Δ :

First Inequality

$$4 r (16 R - 5 r) \leq (\sum a_i)^2 \leq 4 (3 r^2 + 4 r R + 4 R^2). \tag{15}$$

Second Inequality

$$12 r (2 R - r) \leq \sum a_i^2 \leq 4 r^2 + 8 R^2. \tag{16}$$

Third Inequality

$$r^3 (16 R - 5 r) \leq \Delta^2 \leq r^2 (3 r^2 + 4 r R + 4 R^2). \tag{17}$$

Indeed, the first inequality is given by (7) and (14), the second follows from the first and from relation (3), while the third is deduced from the first by the identity $2 \Delta = r \sum a_i$. In all three, equality holds only when the triangle is equilateral.

3. Two Applications

3.1. F. LEUENBERGER has proved [3] the following inequality:

$$18 r R \leq \sum_{i < j} a_i a_j \leq 9 R^2.$$

Using identity (1) and inequality (16), we obtain the stronger result

$$4 r (5 R - r) \leq \sum_{i < j} a_i a_j \leq 4 (r + R)^2. \quad (18)$$

This can be combined with the identity

$$\sum_{i < j} a_i a_j = 2 R \sum h_i,$$

where h_i ($i = 1, 2, 3$) are the altitudes of T , to obtain the inequality

$$\frac{2 r (5 R - r)}{R} \leq \sum h_i \leq \frac{2 (r + R)^2}{R}. \quad (19)$$

3.2. Another result due to the same author [4] is

$$\frac{\sqrt{3}}{R} \leq \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq \frac{\sqrt{3}}{2 r}.$$

Here it is possible, by applying (18), to sharpen the left side. For we have

$$\sum \frac{1}{a_i} = \frac{\sum_{i < j} a_i a_j}{\prod a_i},$$

and by the inequality between arithmetic and geometric means,

$$\frac{1}{3} \sum_{i < j} a_i a_j \geq (\prod a_i)^{2/3},$$

whence

$$(\prod a_i)^{-1} \geq \sqrt[3]{27} \left(\sum_{i < j} a_i a_j \right)^{-3/2}.$$

Multiplying both sides by $\sum_{i < j} a_i a_j$, we get

$$\sum \frac{1}{a_i} \geq \left(\frac{27}{\sum_{i < j} a_i a_j} \right)^{1/2},$$

and it follows from (18) that

$$\frac{3 \sqrt{3}}{2 (r + R)} \leq \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq \frac{\sqrt{3}}{2 r}. \quad (20)$$

4. Corollaries

Our three inequalities can be used to obtain some well-known results. We shall mention three such examples.

4.1. Inequality (16), together with the classical inequality $2 r \leq R$, gives us KUBOTA'S inequality [5]:

$$36 r^2 \leq \sum a_i^2 \leq 9 R^2.$$

4.2. We can write the right side of (17) as

$$\Delta^2 \leq r^2 \left[\left(4 R^2 + \frac{8}{3} r R + \frac{1}{3} r^2 \right) + \frac{1}{3} (4 r R + 8 r^2) \right].$$

Since $4 r^2 \leq 2 r R \leq R^2$, the value of the second parenthesis never exceeds $4 R^2$, and therefore

$$\sqrt{3} \Delta \leq r (4 R + r). \tag{21}$$

But because of identity (3), this is equivalent to

$$4\sqrt{3} \Delta \leq (a_1 + a_2 + a_3)^2 - 2 (a_1^2 + a_2^2 + a_3^2),$$

or

$$4\sqrt{3} \Delta \leq a_1^2 + a_2^2 + a_3^2 - [(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_1)^2],$$

an inequality which was first proved by P. FINSLER and H. HADWIGER [6].

4.3. Finally, since

$$\prod a_i = \frac{\sum_{i < j} a_i a_j}{\sum 1/a_i},$$

we have by (18) and (20)

$$\left(\frac{\sqrt{3}}{2}\right)^3 \cdot \prod a_i \leq (r + R)^3 \leq \left(\frac{3 R}{2}\right)^3,$$

and therefore

$$\Delta = \frac{a_1 a_2 a_3}{4 R} \leq \frac{\sqrt{3}}{4} (a_1 a_2 a_3)^{2/3},$$

whence the chain of inequalities

$$(\sqrt{3} \Delta)^{3/2} \leq \left(\frac{\sqrt{3}}{2}\right)^3 a_1 a_2 a_3 \leq (r + R)^3, \tag{22}$$

parts of which have already been pointed out by L. CARLITZ [7].

In concluding, we note that although (20) and (22) might lead one to suspect that there exists an inequality between $\sum a_i$ and $2 \sqrt{3} (r + R)$, this is not the case, as the following example shows:

For $a_1 = 3$, $a_2 = 4$, $a_3 = 5$, we have $r = 1$ and $R = 5/2$, and therefore $\sum a_i < 2 \sqrt{3} (r + R)$, while in the isoceles triangle $a_1 = 2$, $a_2 = a_3 = 6$, we have $r = \sqrt{35}/7$ and $R = 18/\sqrt{35}$, so that $\sum a_i > 2 \sqrt{3} (r + R)$. J. STEINIG, Zürich

REFERENCES

[1] E. W. HOBSON, *A Treatise on Plane Trigonometry* (Cambridge University Press, 1928) p. 200, (18).
 [2] Lösung zur Aufgabe 266, *El. Math.* 12, 65 (1957).
 [3] F. LEUENBERGER, *Einige Dreiecksungleichungen*, *El. Math.* 13, 121–126 (1958).
 [4] F. LEUENBERGER, *Dreieck und Viereck als Extremalpolygone*, *El. Math.* 15, 77–79 (1960).
 [5] T. KUBOTA, *Einige Ungleichheiten für das Dreieck und das konvexe Polygon*, *Tôhoku Math. J.* 25, 122–126 (1925).
 [6] P. FINSLER und H. HADWIGER, *Einige Relationen im Dreieck*, *Comm. Math. Helv.* 10, 316–326 (1938).
 [7] L. CARLITZ, Problem E 1454, *Amer. Math. Monthly* 68, 177 (1961).