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Concurrencies and Areas in a Triangle

It is astonishing but true (see e.g., [1]¹⁾) that many elementary results relating to the triangle are still being discovered. The following note gives a fresh derivation of some known results, along with simple extensions, some of these being applicable to various other plane configurations.

We begin with a proof of the generalized DUDENEY-STEINHAUS theorem (see [2]):

In a triangle ABC , transversals²⁾ AX , BY , CZ are drawn from the vertices cutting the opposite sides at points X , Y , Z dividing these sides internally in the respective ratios $s : (1 - s)$, $t : (1 - t)$, $u : (1 - u)$. These transversals meet in pairs at L , M , N (see Fig. 1). Then

$$\frac{\Delta LMN}{\Delta ABC} = \frac{\{(1-s)(1-t)(1-u) - stu\}^2}{(st+1-t)(tu+1-u)(us+1-s)} \left(\frac{BX}{BC} = s, \frac{CY}{CA} = t, \frac{AZ}{AB} = u \right).$$

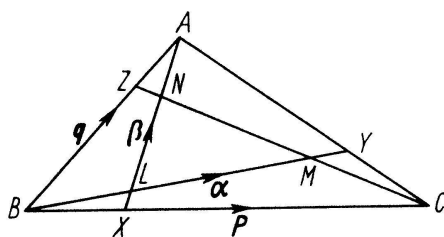


Figure 1

We put $\vec{BC} = \mathbf{p}$, $\vec{BA} = \mathbf{q}$, $\vec{LM} = \boldsymbol{\alpha}$, $\vec{LN} = \boldsymbol{\beta}$. Then, $\vec{AX} = s\mathbf{p} - \mathbf{q}$, $\vec{BY} = (1-t)\mathbf{p} + t\mathbf{q}$, $\vec{CZ} = (1-u)\mathbf{q} - \mathbf{p}$. Writing $\lambda \vec{BY} = \vec{BL} = s\mathbf{p} + \mu \vec{XA}$, we have

$$\lambda(1-t)\mathbf{p} + \lambda t\mathbf{q} = s(1-\mu)\mathbf{p} + \mu\mathbf{q},$$

¹⁾ Numbers in brackets refer to References, page 55.

²⁾ Other writers refer to these lines through the vertices as *cevians*, *cevians*, *radials*, and *nedians*; see, e.g., the references to *School Science and Mathematics* at the end of this paper.

whence $\lambda(1-t) = s(1-\mu)$ and $\lambda t = \mu$, so that $\lambda = s/(st+1-t)$, $\mu = st/(st+1-t)$. From these, and similar, results it follows immediately that

$$\left. \begin{aligned} \frac{BL}{BY} &= \frac{s}{st+1-t}, & \frac{XL}{XA} &= \frac{st}{st+1-t} \\ \frac{CM}{CZ} &= \frac{t}{tu+1-u}, & \frac{YM}{YB} &= \frac{tu}{tu+1-u} \\ \frac{AN}{AX} &= \frac{u}{us+1-s}, & \frac{ZN}{ZC} &= \frac{us}{us+1-s} \end{aligned} \right\} \quad (1)$$

so that

$$\alpha = \vec{BY} - (\vec{BL} + \vec{MY}) = \left\{1 - \frac{BL}{BY} - \frac{MY}{BY}\right\} \vec{BY} = \phi \frac{t\mathbf{q} + (1-t)\mathbf{p}}{(st+1-t)(tu+1-u)},$$

where $\phi = (1-s)(1-t)(1-u) - stu$; similarly

$$\beta = \phi \frac{\mathbf{q} - s\mathbf{p}}{(st+1-t)(us+1-s)}.$$

Thus, as $st+1-t > 0$, etc.,

$$\begin{aligned} \frac{\Delta LMN}{\Delta ABC} &= \left| \frac{\alpha \times \beta}{\mathbf{p} \times \mathbf{q}} \right| = \frac{\phi^2 |\{t\mathbf{q} + (1-t)\mathbf{p}\} \times \{\mathbf{q} - s\mathbf{p}\}|}{(st+1-t)^2 (tu+1-u)(us+1-s) |\mathbf{p} \times \mathbf{q}|} \\ &= \frac{\{(1-s)(1-t)(1-u) - stu\}^2}{(st+1-t)(tu+1-u)(us+1-s)} = f(s, t, u), \text{ say.} \end{aligned} \quad (2)$$

Among the consequences of the result (2) we may mention the following:

I. Taking $s = t = u = 1/3$ we obtain the DUDENEY-STEINHAUS theorem: if X, Y, Z divide the sides of ΔABC , in cyclic order, in the ratio 1:2 then $\Delta LMN = (1/7) \Delta ABC$.

II. Clearly, $\Delta LMN = 0$ if, and only if, $f(s, t, u) = 0$, i.e.

$$(1-s)(1-t)(1-u) = stu, \quad (3)$$

($0 \leq s, t, u \leq 1$). As (3) holds for $s = t = u = 1/2$, it follows immediately that the medians of ΔABC are concurrent.

III. Taking $s = t = u = 1/2$, it follows readily from the relations (1) that the medians of a triangle divide each other in the ratio 2:1. The converse result, that AX, BY, CZ are the medians of ΔABC if they are concurrent and divide each other in the ratio 2:1, is moderately difficult to prove by elementary methods; however, using the relations (1), a simple calculation shows that, in this case, $s = t = u = 1/2$.

IV. For $0 < s, t, u < 1$,

$$\frac{BX}{XC} \frac{CY}{YA} \frac{AZ}{ZB} = \frac{stu}{(1-s)(1-t)(1-u)}.$$

As the transversals AX, BY, CZ are concurrent if, and only if, $(1-s)(1-t)(1-u) = stu$, CEVA's theorem and its converse follow immediately.

V. Some interesting results arise from the case in which the transversals divide the sides, in cyclic order, in the same ratio. Suppose $BX : XC = CY : YA = AZ : ZB = \lambda : \mu$; then, $s = t = u = \lambda/(\lambda + \mu)$, and

$$\frac{\Delta LMN}{\Delta ABC} = \frac{(\lambda - \mu)^2}{\lambda^2 + \lambda\mu + \mu^2}. \quad (4)$$

(It will be clear that the transversals now cut, in cyclic order, in the same ratio.) We consider $\triangle XYZ$ (the MENELAIC triangle [3]): with the above notation,

$$\vec{XY} = (1 - s) \mathbf{p} + t (\mathbf{q} - \mathbf{p}), \quad \vec{XZ} = (1 - u) \mathbf{q} - s \mathbf{p},$$

whence

$$\frac{\triangle XYZ}{\triangle ABC} = \frac{|\vec{XY} \times \vec{XZ}|}{|\mathbf{p} \times \mathbf{q}|} = (1 - s)(1 - t)(1 - u) + stu;$$

Thus, in the case considered,

$$\frac{\triangle XYZ}{\triangle ABC} = \frac{\lambda^2 - \lambda\mu + \mu^2}{(\lambda + \mu)^2} \quad (5)$$

and

$$\frac{\triangle LMN}{\triangle XYZ} = \frac{(\lambda^2 - \mu^2)^2}{\lambda^4 + \lambda^2\mu^2 + \mu^4}. \quad (6)$$

The expressions on the right in (4), (5), (6) are symmetric in λ and μ , so that interchange of λ, μ leaves the corresponding ratios invariant; it is easily seen that, in general, the effect of this interchange is not such as to transform the triangles XYZ , LMN into identical triangles.

VI. It is easily shown that, when one or more of the points X, Y, Z divide the corresponding sides of triangle ABC *externally*, the ratio of the areas of the triangles XYZ, ABC is given by

$$\frac{\triangle XYZ}{\triangle ABC} = |(1 - s)(1 - t)(1 - u) + stu|.$$

Accordingly, $stu/\{(1 - s)(1 - t)(1 - u)\} = -1$ (here, we suppose X, Y, Z do not coincide with any vertex of $\triangle ABC$), if and only if $\triangle XYZ = 0$; i.e., if, and only if, X, Y, Z are collinear. Thus we obtain MENELAUS' theorem and its converse. (For this case it follows from Pasch's axiom that at least one of the points X, Y, Z lies outside $\triangle ABC$.)

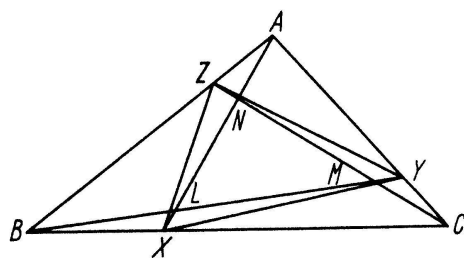


Figure 2

VII. By expressing $\vec{AN}, \vec{ZN}, \vec{MY}, \vec{CM}$, etc., in terms of $\mathbf{p}, \mathbf{q}$ we can similarly calculate the ratios $\triangle ANZ : \triangle ABC$, $\triangle ANC : \triangle ABC$, $\triangle BXL : \triangle BCY$, etc., together with ratios of areas of quadrilaterals such as $LXCM : ANMY$, and so on. If X, Y, Z divide corresponding sides of triangle ABC in the same ratio, then $\triangle ANZ = \triangle BLX = \triangle CMY$, and the quadrilaterals $LXCM, MYAN, NZBL$ are of equal area.

VIII. If AX, BY, CZ are the interior bisectors of the corresponding angles of the triangle ABC , then, with a common notation ($BC = a$, etc.), we have $s/(1 - s) = c/b$,

$t/(1-t) = a/c$, $u/(1-u) = b/a$, whence $(1-s)(1-t)(1-u) = stu$; accordingly, these interior bisectors meet at a point I (the incentre). More generally, if AX , BY , CZ meet at an interior point J , then by (1)

$$\frac{AJ}{JX} = \frac{u}{(1-u)(1-s)}, \quad \frac{BJ}{JY} = \frac{s}{(1-s)(1-t)}, \quad \frac{CJ}{JZ} = \frac{t}{(1-t)(1-u)},$$

and $stu = (1-s)(1-t)(1-u)$, so that

$$\frac{AJ}{JX} \frac{BJ}{JY} \frac{CJ}{JZ} = \frac{1}{stu};$$

in particular, if J is the incentre I of triangle ABC , then we obtain

$$\frac{AI}{IX} \frac{BI}{IY} \frac{CI}{IZ} = \frac{(a+b)(b+c)(c+a)}{abc}.$$

It will be clear that a similar result can be obtained for the case in which J lies at the centre of an escribed circle.

IX. If AX , BY , CZ are the altitudes of triangle ABC then

$$\frac{s}{1-s} = \frac{c \cos B}{b \cos C}, \quad \frac{t}{1-t} = \frac{a \cos C}{c \cos A}, \quad \frac{u}{1-u} = \frac{b \cos A}{a \cos B} \quad (7)$$

so that we again have $stu = (1-s)(1-t)(1-u)$; this establishes concurrency of the altitudes. From this result and the result obtained in V we see that the area of the pedal triangle XYZ is given by

$$\frac{\Delta XYZ}{\Delta ABC} = |(1-s)(1-t)(1-u) + stu| = 2|stu|;$$

hence, by (7), $\Delta XYZ = 2|\cos A \cos B \cos C| \Delta ABC$.

X. Now let Y , Z divide CA , AB in the respective ratios $p:q$ and $q:p$. As p/q varies, the locus of the intersection of BY , CZ is the median through A . This result follows immediately from CEVA's theorem.

XI. Let a , b , c denote the lengths of the sides of triangle ABC and let m_1 , m_2 , m_3 denote the lengths of the transversals AX , BY , CZ . Put $\psi = (m_1^2 + m_2^2 + m_3^2)/(a^2 + b^2 + c^2)$. It is well known that, if AX , BY , CZ are medians, then

$$\psi = \frac{3}{4}. \quad (8)$$

More generally, if X , Y , Z divide BC , CA , AB in the same ratio p/q ($0 \leq p/q < \infty$), it is easily shown that

$$\psi = \frac{p^2 + pq + q^2}{(p+q)^2} \leq 1;$$

For $0 \leq p/q < \infty$, $\inf(\psi) = 3/4$, the infimum being attained for $p = q$ (i.e. for the case in which AX , BY , CZ are medians).

Now suppose X , Y , Z divide the sides of triangle ABC in the respective ratios $s/(1-s)$, $t/(1-t)$, $u/(1-u)$. We can readily show that

$$\begin{aligned} m_1^2 + m_2^2 + m_3^2 &= a^2(s^2 + 1 + u - s - t) + b^2(t^2 + 1 + s - t - u) \\ &\quad + c^2(u^2 + 1 + t - u - s); \end{aligned}$$

and from this it follows that, for $-\infty < s, t, u < +\infty$,

$$\inf(\psi) = \frac{3}{4} - \frac{1}{4(a^2 + b^2 + c^2)} \left\{ \frac{(a^2 - b^2)^2}{c^2} + \frac{(b^2 - c^2)^2}{a^2} + \frac{(c^2 - a^2)^2}{b^2} \right\} \quad (9)$$

this infimum being attained for

$$s = \frac{c \cos B}{a}, \quad t = \frac{a \cos C}{b}, \quad u = \frac{b \cos A}{c}.$$

(This last result is immediate when we note that ψ attains its least value when AX , BY , CZ are altitudes of triangle ABC .)

Restricting $0 \leq s, t, u \leq 1$, i.e. all transversals internal, we have

Theorem 1. If m_1, m_2, m_3 are the lengths of transversals drawn from the vertices A, B, C of an acute-angled triangle, to the opposite sides of lengths a, b and c and if

$$\psi = \frac{m_1^2 + m_2^2 + m_3^2}{a^2 + b^2 + c^2},$$

then $1/2 < \text{Min}\{\psi\} \leq 3/4$, the upper bound being attained in an equilateral triangle.

Proof:

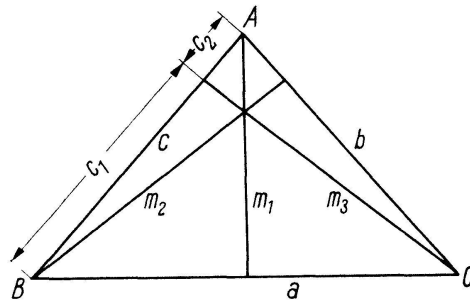


Figure 3

If all angles $A, B, C \leq 90^\circ$ set $a \geq b \geq c$. Therefore $90^\circ \geq A \geq B \geq C$ and $A \geq B \geq 45^\circ$, m_1, m_2, m_3 are altitudes.

$$m_3^2 = a^2 - c_1^2, \quad m_2^2 = b^2 - c_2^2$$

therefore

$$2m_3^2 = a^2 + b^2 - (c_1^2 + c_2^2)$$

but $c^2 > c_1^2 + c_2^2$ and $2m_3^2 > a^2 + b^2 - c^2$.

$$m_2 = c \sin A \geq \frac{c}{\sqrt{2}}, \quad \text{also } m_1 \geq \frac{c}{\sqrt{2}}$$

therefore

$$2m_1^2 + 2m_2^2 + 2m_3^2 > a^2 + b^2 + c^2,$$

thus

$$\text{Min}\{\psi\} > \frac{1}{2}.$$

$\text{Min}\{\psi\}$ may be made to differ from $1/2$ by as little as we please as may be seen in an acute-angled isosceles triangle whose base is arbitrarily small.

Again, from (9) we see that $\text{Min}\{\psi\} \geq 3/4$ and if $a = b = c$, $\text{Min}\{\psi\} = 3/4$.

Theorem 2. With the previous notation, if ABC is an acute-angled triangle,

$$1 \leq \text{Max} \{\psi\} < 3/2.$$

Proof:

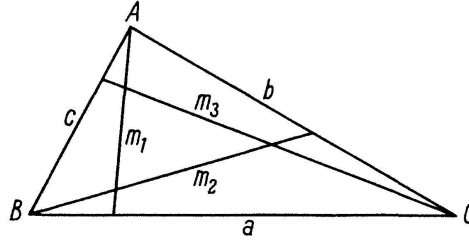


Figure 4

Let $a \geq b \geq c$, normalize side a to unity, thus $1 \geq b \geq c$. Then

$$m_1 \leq b, \quad m_2 \leq 1, \quad m_3 \leq 1,$$

and

$$\psi \leq \frac{2 + b^2}{1 + b^2 + c^2} \text{ for fixed } b \text{ and } c.$$

Assume

$$\frac{2 + b^2}{1 + b^2 + c^2} \geq \frac{3}{2}$$

then we have $1 \geq b^2 + 3c^2 > b^2 + c^2$, but by hypothesis $1 < b^2 + c^2$, thus the contradiction yields

$$\text{Max} \{\psi\} < \frac{3}{2}.$$

However, the maximum may differ from $3/2$ by as little as we please, as may be seen in an acute-angled isosceles triangle whose base is arbitrarily small.

Again, keeping b fixed, the denominator of $(2 + b^2)/(1 + b^2 + c^2)$ is largest when $c = b$, thus we require to minimize $f(b) = (2 + b^2)/(1 + 2b^2)$ under the restriction $0 \leq b \leq 1$.

$f(b)$ being a decreasing function has its minimum value at the maximum value of b , thus $f(b)_{\min} = 1$ and $\text{Max} \{\psi\} \geq 1$ exhibiting equality in an equilateral triangle.

Theorem 3. With the previous notation, if ABC is an obtuse-angled triangle,

$$\frac{1}{3} < \psi < \frac{3 + \sqrt{3}}{3}.$$

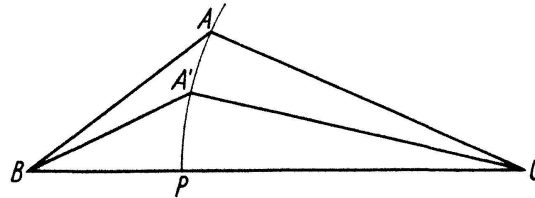


Figure 5

Normalizing a to unity, set $1 \geq b \geq c$ and then

$$m_1 \leq b, \quad m_2 \leq 1, \quad m_3 \leq 1,$$

and $\psi \leq (2 + b^2)/(1 + b^2 + c^2)$ for fixed b and c .

With centre C and radius b , describe an arc of a circle through vertex A cutting BC at P . Join C and B to A' , any point on the arc AP inside the triangle ABC .

In $\triangle A'BC$ angle $A' > \text{angle } A$, thus $BA' < BA = c$ and

$$\frac{2 + b^2}{1 + b^2 + BA'^2} > \frac{2 + b^2}{1 + b^2 + c^2}.$$

Thus we may obtain obtuse-angled triangles $A'BC$ in which the quantity $(2 + b^2)/(1 + b^2 + c^2)$ becomes progressively larger. $\text{Sup} \{(2 + b^2)/(1 + b^2 + c^2)\}$ occurs for $c = 1 - b$, and $(2 + b^2)/(1 + b^2 + (1 - b)^2)$ becomes largest for $b = \sqrt{3} - 1$. Hence

$$\text{Sup} \left\{ \frac{2 + b^2}{1 + b^2 + c^2} \right\} = \frac{3 + \sqrt{3}}{3}$$

and

$$\psi < \frac{3 + \sqrt{3}}{3},$$

the difference $|\psi - (3 + \sqrt{3})/3|$ being arbitrarily small in the triangle whose sides approach 1, $\sqrt{3} - 1$, $2 - \sqrt{3}$ respectively.

Again, in $\triangle ABC$

$$m_2 \geq c, \quad m_3 \geq b,$$

and $a < b + c$ (reverting to side $BC = a$).

Thus

$$m_1^2 + m_2^2 + m_3^2 > b^2 + c^2. \quad (10)$$

Further, $a^2 < 2b^2 + 2c^2$ as $2bc < b^2 + c^2$, therefore

$$a^2 + b^2 + c^2 < 3(b^2 + c^2). \quad (11)$$

From (10) and (11) therefore $(m_1^2 + m_2^2 + m_3^2)/(a^2 + b^2 + c^2) > 1/3$, the difference $|\psi - 1/3|$ being arbitrarily small in an obtuse-triangle whose sides approach a , a , $2a$ respectively.

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