Two non-negative quadratic forms

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und daher

$$\varphi(U \cup W)\, f\,((U + \lambda t) \cup W) = \varphi(U)\,(f(U) + \lambda t) + \varphi(W)\, f(W)$$
$$= \varphi(U \cup W)\, f(U \cup W) + \lambda \varphi(U) t$$
$$\notin H^+$$

für alle grossen λ. Wegen

$$f\,((U + \lambda t) \cup W) \in \operatorname{konv}\,((U + \lambda t) \cup W) \subset H^+$$

muss also $\varphi(U \cup W) < 0$ sein. Andererseits gilt für hinreichend grosse λ

$$\varphi(U \cup W)\, f(U \cup (W + \lambda t)) = \varphi(U \cup W)\, f(U \cup W) + \lambda t \in H^+$$

und

$$f(U \cup (W + \lambda t)) \in \operatorname{konv}\,(U \cup (W + \lambda t)) \subset H^+ ,$$

was $\varphi(U \cup W) > 0$ nach sich zieht. Die Annahme war also falsch, das heisst φ ist definit.

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Two Non-Negative Quadratic Forms

I. Introduction

In problem E 2348 [1], L. Carlitz has given the inequality

$$\sum R_1\,(r_2 + r_3) \geq \sum (r_1 + r_2)\,(r_1 + r_3) \tag{1}$$

where R_1, R_2, R_3 and r_1, r_2, r_3 denote the distances from an interior point of a triangle ABC to the vertices A, B, C and the sides a, b, c, respectively. Coupling (1) with the known lower bounds $R_1 \geq (r_2\,c + r_3\,b)/a$, etc. [2, p. 107], suggests the stronger inequality

$$\left.\begin{aligned} F_1 \equiv \left\{\frac{b}{c} + \frac{c}{b} - 1\right\} r_1^2 + \left\{\frac{c}{a} + \frac{a}{c} - 1\right\} r_2^2 + \left\{\frac{a}{b} + \frac{b}{a} - 1\right\} r_3^2 \\ - \left\{3 - \frac{b+c}{a}\right\} r_2\,r_3 - \left\{3 - \frac{c+a}{b}\right\} r_3\,r_1 - \left\{3 - \frac{a+b}{c}\right\} r_1\,r_2 \geq 0\,. \end{aligned}\right\} \tag{2}$$

We shall show that $F_1 \geq 0$ is indeed valid for all triangles ABC and *any* real values of r_1, r_2, r_3. Then by using the lower bounds $b/c + c/b \geq 2$, etc., inequality (2) can be partially strengthened to

$$\left.\begin{aligned} F_2 \equiv r_1^2 + r_2^2 + r_3^2 - \left\{3 - \frac{b+c}{a}\right\} r_2 r_3 - \left\{3 - \frac{c+a}{b}\right\} r_3 r_1 \\ - \left\{3 - \frac{a+b}{c}\right\} r_1 r_2 \geq 0\,. \end{aligned}\right\} \quad (3)$$

However, here r_1, r_2, r_3 are to be arbitrary non-negative numbers.

II. $\mathbf{F_1 \geq 0}$

A standard way of showing F_1 is non-negative is to show that its associated matrix

$$M = \left\| \begin{matrix} \dfrac{b^2 - bc + c^2}{bc} & \dfrac{a + b - 3c}{2c} & \dfrac{a + c - 3b}{2b} \\ \dfrac{b + a - 3c}{2c} & \dfrac{c^2 - ca + a^2}{ca} & \dfrac{b + c - 3a}{2a} \\ \dfrac{c + a - 3b}{2b} & \dfrac{c + b - 3a}{2a} & \dfrac{a^2 - ab + b^2}{ab} \end{matrix} \right\|$$

is positive semidefinite. As is well known, F_1 is a non-negative form *iff* all the principal minors of M are ≥ 0. The first two leading ones, M_1 and M_2 are easy to establish. For

$$bc\, M_1 = (b - c)^2 + bc > 0\,,$$

and after some manipulation

$$4abc^2 M_2 = 4c^2 \{\textstyle\sum a^2 - \sum ab\} + ab\{2 \sum ab - \sum a^2\}\,.$$

That M_2 is non-negative, follows from two elementary triangle inequalities [2, p. 11]. The non-negativity of the remaining 1st and 2nd order principal minors follows by symmetry.

To simplify the valuation of $M_3 = \det M$, we make the duality transformation [3],

$$a = y + z\,, \quad b = z + x\,, \quad c = x + y$$

where x, y, z are arbitrary non-negative numbers, not all zero. After some simple algebra, we obtain

$$(x + y)^2 (y + z)^2 (z + x)^2 M_3 = \left| \begin{matrix} P - 2yz & (z - x - y)\,(z + x) & (y - z - x)\,(y + x) \\ (z - x - y)\,(x + y) & P - 2zx & (x - y - z)\,(x + y) \\ (y - z - x)\,(y + z) & (x - y - z)\,(x + z) & P - 2xy \end{matrix} \right|$$

where $P = \Sigma x^2 + \Sigma xy$. We now add row 2 and row 3 to row 1, giving a row of constant terms, $\Sigma x^2 - \Sigma xy$, which can be factored out. Next, we subtract column 2 form column 3 and then column 1 from column 2, leading to a 2×2 determinant. Then adding the rows together, we finally obtain

$$(x+y)^2(y+z)^2(z+x)^2 M_3 = \{\sum xy\}\{\sum x^2 - \sum xy\}\{\sum x^2 + 3\sum xy\} \geq 0$$

with equality *iff* $x = y = z$ or two of x, y, z are zero. Consequently, (1) and (2) are valid with equality *iff* ABC is equilateral (we are excluding degenerate triangles).

III. $F_2 \geq 0$

By just considering the case $r_3 = 0$, $r_1 r_2 < 0$ and $(a+b)/c$ large, it follows that inequality (3) is not valid for all real r_1, r_2, r_3. Consequently, we restrict r_1, r_2, r_3 to non-negative values and replace them by x^2, y^2, z^2, respectively. Also, we replace a, b, c by $q+r$, $r+p$, $p+q$, respectively, where p, q, r are arbitrary non-negative numbers. Inequality (3) now takes the form

$$F_2' \equiv x^4 + y^4 + z^4 - 2uy^2z^2 - 2vz^2x^2 - 2wx^2y^2 \geq 0 \qquad (3)'$$

where

$$u = 1 - \frac{p}{q+r}, \quad v = 1 - \frac{q}{r+p}, \quad w = 1 - \frac{r}{p+q}.$$

Since F_2' is a biquadratic in x, y, z, it follows by a theorem of Hilbert [4] that if $F_2' \geq 0$, then it can be expressed as the sum of squares of three real polynomials and conversely. Consequently, our proof of (3)′ is based on exhibiting such a representation.

Without loss of generality, we can assume that $p \geq q \geq r$. Whence, $u < 1$, $0 \leq v, w \leq 1$. F_2' can now be expressed in the form

$$F_2' = \{x^2 - vz^2 - wy^2\}^2 + \{y^2(1-w^2)^{1/2} - z^2(1-v^2)^{1/2}\}^2 + 2Gy^2z^2$$

where

$$G = (1-v^2)^{1/2}(1-w^2)^{1/2} - u - vw.$$

In order to show that G is non-negative, we consider, separately, three intervals for u, i.e., (C_1) $u \leq u_0$, (C_2) $0 \leq u \leq 1$ and (C_3) $u_0 < u < 0$. The first case (C_1) is the easiest since u_0 is the negative root of a certain cubic such that $u + vw \leq 0$. For (C_2), p, q, r are possible sides of a triangle. Thus, we can make the duality transformation $p = \beta + \gamma$, $q = \gamma + \alpha$, $r = \alpha + \beta$ where $\alpha, \beta, \gamma \geq 0$. On squaring out $G = 0$, we get

$$1 \geq u^2 + v^2 + w^2 + 2uvw. \qquad (4)$$

After a considerable amount of simple algebra, (4) can be expressed in the form

$$\left.\begin{aligned} &\frac{7}{3}T_1^2T_2(T_1^2 - 3T_2) + \frac{4}{9}T_1^3(T_1T_2 - 9T_3) + \frac{10}{9}T_1T_2(T_1^3 - 27T_3) \\ &\qquad + \frac{1}{9}(T_1^3 \cdot T_1T_2 - 27T_3 \cdot 9T_3) \geq 0 \end{aligned}\right\} \qquad (5)$$

where

$$T_1 = \alpha + \beta + \gamma\,, \quad T_2 = \beta\gamma + \gamma\alpha + \alpha\beta\,, \quad T_3 = \alpha\beta\gamma\,.$$

Since it is well known that

$$T_1^2 \geq 3\,T_2\,, \quad T_1^3 \geq 27\,T_3\,, \quad T_1\,T_2 \geq 9\,T_3\,,$$

inequality (5) is valid and with equality *iff* $\alpha = \beta = \gamma$.

For the last case (C_3), we resort to a geometric proof. First, we note that u, v, w satisfy the identity

$$\begin{vmatrix} 1 & u-1 & u-1 \\ v-1 & 1 & v-1 \\ w-1 & w-1 & 1 \end{vmatrix} = 0$$

or equivalently

$$H \equiv \left\{v - \frac{4-3\,u}{3-2\,u}\right\}\left\{w - \frac{4-3\,u}{3-2\,u}\right\} - \left\{\frac{2-u}{3-2\,u}\right\}^2 = 0\,. \tag{6}$$

For a fixed u, permissible values of v, w will then lie on the lower branch of the hyperbola $H = 0$ which is contained in the unit square $0 \leq v, w \leq 1$. If we also wish to have $u + vw \leq 0$, then the critical value of u (denoted by u_0) is determined by requiring the lower branch of $H = 0$ to be tangent to the positive branch of $vw = -u$ (see figure). u_0 is then the negative root of $\sqrt{-u} = 2\,(1-u)/(3-2u)$ or $4u^3 - 8u^2 + u + 4 = 0$. Here, $u_0 \approx -.57$.

In order to show that (4) is valid for each fixed u in $(u_0, 0)$, it suffices to show that the part of the lower branch of $H = 0$ lying above the positive branch of $vw = -u$ and within the square $0 \leq v, w \leq 1$, also lies inside the ellipse (see figure)

$$E \equiv v^2 + w^2 + 2\,u\,v\,w - (1 - u^2) = 0\,.$$

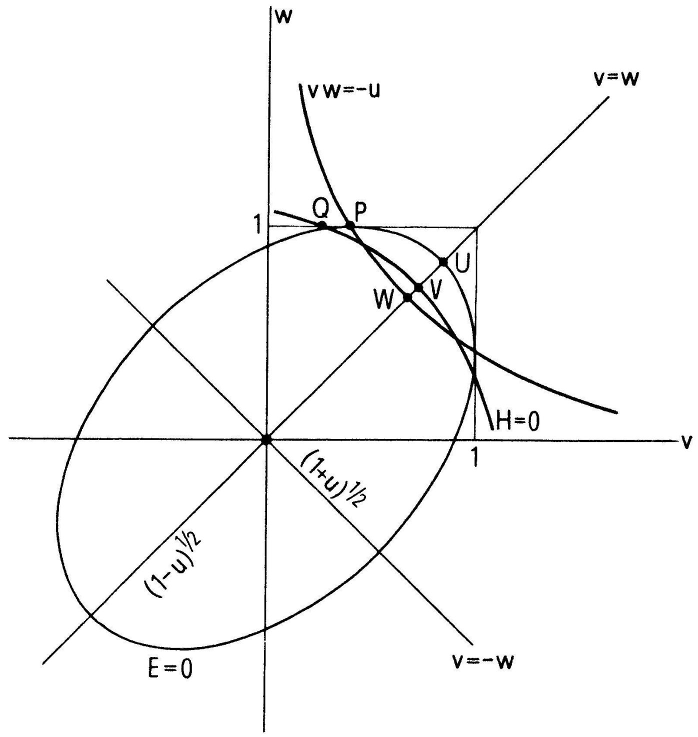
The semi-axes of E lie on the lines $v \pm w = 0$ and their lengths are $\sqrt{1-u}$ and $\sqrt{1+u}$, respectively.
The ellipse is inscribed in the square $-1 \leq v, w \leq 1$. The positive branch of the hyperbola $v\,w = -u$ passes through two points of tangency. The three curves are symmetric with respect to the line $v = w$. The coordinates of the indicated points are given by

$$P = (-u, 1)\,, \quad Q = \left(\frac{u}{u-1}\,, 1\right),$$

$$U_v = \sqrt{(1-u)/2}\,, \quad V_v = \frac{2\,(1-u)}{3-2u}\,, \quad W_v = \sqrt{-u/2}$$

(U_v denotes the abscissa of U, etc.).

Since V lies between W and U and Q is to the left of P, our inequality is established.



In addition to having equality in (3)′ for the case, $a = b = c$, $x = y = z$, we also have equality for a degenerate triangle corresponding to P and Q coinciding at (0,1). For this case, $r = 0$, $x = y$, $a = q$, $b = p$, $c = p + q$.

IV. Triangle inequalities

Numerous triangle inequalities can be obtained from the forms F_1 and F_2 by letting r_1, r_2, r_3 be particular functions of the sides, e.g., $r_1 = a, a^2, ab, b$ etc. (r_2 and r_3 are then chosen by a cylic interchange of a, b, c). Most of these inequalities are not particularly elegant, e.g.,

$$abc \sum a^2 + \sum (a + b)\, a^2 b^2 \geq 3abc \sum ab\,, \tag{7}$$

$$2a^2 b^2 c^2 \sum ab + \sum b^4 c^4 \geq 3a^2 b^2 c^2 \sum c^2\,. \tag{8}$$

However, we can rewrite (3)′ into the more appealing form (geometrically)

$$\frac{p}{q+r}\, b^2 c^2 + \frac{q}{r+p}\, c^2 a^2 + \frac{r}{p+q}\, a^2 b^2 \geq 8\Delta^2 \tag{9}$$

where Δ denotes the area of triangle ABC. It is to be noted that if $|a|$, $|b|$, $|c|$ did not form a triangle, the r.h.s. of (9) would be negative giving a trivial inequality. In terms of angles, (9) is given by

$$\frac{p \csc^2 A}{q+r} + \frac{q \csc^2 B}{r+p} + \frac{r \csc^2 C}{p+q} \geq 2\,. \tag{9$'$}$$

There is equality in (9) and (9)′ *iff* $A = B = C = \pi/3$, $p = q = r$. If we also allow degenerate triangles, there is also equality *iff* $A = B = \pi/2$, $p = q$, $r = 0$ (assuming $p \geq q \geq r$). Inequalities (9) and (9)′ generalize the known special case corresponding to $p = q = r$ [1, pp. 31, 45].

If we now let $(a', b', c') = (a^2, b^2, c^2)$ and restrict ABC to be an acute triangle, then a', b', c' are sides of a general triangle of area Δ'. By virtue of the known inequality, $4\,\Delta^2 \geq \sqrt{3}\,\Delta'$, of Finsler and Hadwiger [1, p. 91] together with (9), gives

$$\frac{p}{q+r}\,b'\,c' + \frac{q}{r+p}\,c'\,a' + \frac{r}{p+q}\,a'\,b' \geq 2\sqrt{3}\,\Delta' \tag{10}$$

or equivalently

$$\frac{p\csc A'}{q+r} + \frac{q\csc B'}{r+p} + \frac{r\csc C'}{p+q} \geq \sqrt{3}\,. \tag{10$'$}$$

The last two forms generalize the known special case corresponding to $p = q = r$ [2, p. 31, 43].

Other related extensions will be given in a subsequent paper. Also, for other examples of non-negative quadratic forms and their associated triangle inequalities, see [5], [6] and the references therein.

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A Note on Discontinuous Functions

Let $\mathfrak{J}$ denote the class of real-valued functions defined and everywhere discontinuous on an interval $[a, b]$. F. Fricker [1] considered questions concerning the set $\mathcal{H}(f) = \{x\colon \lim_{y\to x} f(y) \text{ exists}\}$ for $f \in \mathfrak{J}$. He asked whether it is possible for $\mathcal{H}(f)$ to be dense in $[a, b]$. A negative answer to this question was obtained by R. Jeltsch [2]. The purpose of this note is to characterize those sets H for which there exists $f \in \mathfrak{J}$ such that $H = \mathcal{H}(f)$.

We begin with three lemmas.

Lemma 1. For any real-valued function f defined on $[a, b]$ the set $\mathcal{H}(f) = \{x\colon \lim_{y\to x} f(y) \text{ exists}\}$ is of type G_δ.

Proof. For each $x \in [a, b]$ and $\delta > 0$ let

$$\omega_\delta(x) = \sup\{|\,f(y) - f(z)\,|\colon 0 < |\,y - x\,| < \delta,\ 0 < |\,z - x\,| < \delta\}$$