Kleine Mitteilungen

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$$f(x) = \begin{cases} \frac{1}{k} & \text{if } x = h_k \text{ for some } k \\ \frac{1}{n} & \text{if } x \in A_n \text{ for some } n\,. \\ -\frac{1}{n} & \text{if } x \in B_n \text{ for some } n \end{cases}$$

We show f is everywhere discontinuous and $\lim_{y \to x} f(y)$ exists if and only if $x \in H$. First, suppose $x \in H$. If $x_n \to x$, $x_n \in H$, then $x_n = h_{k_n}$ so $f(x_n) = 1/k_n$ and $\lim_{n \to \infty} f(x_n) = 0$. If $x_n \to x$, $x_n \notin H$, then for each n there exists a natural number q_n such that $x_n \in G_{q_n} - G_{q_n+1}$ so $f(x_n) = 1/q_n$. It is easy to verify that $\lim_{n \to \infty} q_n = \infty$ so $\lim_{n \to \infty} f(x_n) = 0$. It follows that $\lim_{y \to x} f(y) = 0$ for all $x \in H$. Since $f(x) = 1/k$ for some k, f is discontinuous at x.

Now suppose $x \notin H$. There exists a natural number n such that $x \in G_n - G_{n+1}$. But the sets A_n and B_n are each dense in $G_n - G_{n+1}$ so that, by the definition of f, the numbers $1/n$ and $-1/n$ are both in the cluster set of f at x. It follows that $\lim_{y \to x} f(y)$ does not exist.

This completes the proof of the theorem.

Remark 1: The foregoing proof can be easily modified to apply to nowhere continuous functions on a complete separable metric space which is dense in itself.

Remark 2: Since a denumerable set of type G_δ is nowhere dense (in fact, nowhere dense-in-itself), we see that the question posed by F. Fricker has a negative answer.

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Kleine Mitteilungen

There is no Odd Super Perfect Number of the Form $p^{2\alpha}$

In [4] the author defined super perfect numbers as positive integers n such that $\sigma(\sigma(n)) = 2n$, where $\sigma(n)$ denotes the sum of all the positive divisors of n. It has been shown in [4] that an even integer n is super perfect if and only if $n = 2^r$, where $2^{r+1} - 1$ is a prime and posed the existence of odd super perfect numbers as a problem. This is still an open problem. In [2] H. J. Kanold has shown that if n is an odd super perfect number, then n must be a square. In [1] P. Bundschuh posed the problem,

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viz., p^2 (p odd prime) is not a super perfect number. A solution of this problem has been given by H. G. Niederreiter in [3].

The object of the present note is to establish the following:

Theorem. There is no odd super perfect number of the form $p^{2\alpha}$.

Proof: Suppose $p^{2\alpha}$ (p odd prime) is a super perfect number. Then we have

$$\sigma(1 + p + \cdots + p^{2\alpha}) = 2p^{2\alpha}.$$

Let

$$1 + p + \cdots + p^{2\alpha} = \prod_{j=1}^{s} q_j^{\beta_j}. \tag{1}$$

where $q_j' s$ are distinct odd primes.
Then

$$\prod_{j=1}^{s} \sigma(q_j^{\beta_j}) = 2p^{2\alpha} \tag{2}$$

Now, s must be ≥ 2. For, if $s = 1$, then from (1) and (2), it follows that

$$1 + p + \cdots + p^{2\alpha} = q_1^{\beta_1}$$

and

$$1 + q_1 + \cdots + q_1^{\beta_1} = 2p^{2\alpha}.$$

Hence

$$q_1^{\beta_1} \equiv 1 \pmod{p}, \ \beta_1 \text{ is odd and } q_1^{\beta_1} + 1 \equiv 1 \pmod{p},$$

so that $q_1 \equiv 1 \pmod{p}$. Since β_1 is odd, $(1 + q_1) \mid \sigma(q_1^{\beta_1})$, so that $1 + q_1 = 2p^\gamma$, where $1 \leq \gamma \leq 2\alpha$.
Hence $q_1 \equiv -1 \pmod{p}$, which together with $q_1 \equiv 1 \pmod{p}$ implies that $p \mid 2$. This is impossible, since p is an odd prime.

Now, from (2), it follows that exactly one $\sigma(q_j^{\beta_j})$ is even and the rest are odd. Let us suppose without loss of generality that $\sigma(q_1^{\beta_1})$ is even and $\sigma(q_j^{\beta_j})$ is odd for $j = 2, 3, \ldots s$. Hence it follows that β_1 is odd and $\beta_2, \beta_3, \ldots \beta_s$ are all even.

Again from (2), we have

$$(1 + q_1)\left[1 + q_1^2 + \cdots + (q_1^2)^{\frac{\beta_1 - 1}{2}}\right] \prod_{j=2}^{s} \sigma(q_j^{\beta_j}) = 2p^{2\alpha},$$

so that

$$1 + q_1 = 2p^\gamma, \text{ where } 1 \leq \gamma \leq 2\alpha \tag{3}$$

and

$$1 + q_1^2 + \cdots + (q_1^2)^{\frac{\beta_1 - 1}{2}} = p^\delta, \text{ where } 0 \leq \delta \leqslant 2\alpha - \gamma. \tag{4}$$

Furthermore

$$1 + q_j + q_j^2 + \cdots + q_j^{\beta_j} = p^{\gamma_j}, \tag{5}$$

where, $1 \leq \gamma_j \leq 2\alpha - \gamma - \delta$ for $j = 2, 3, \ldots s$.

From (3) we have $q_1 \equiv -1 \pmod p$ and since β_1 is odd, it follows that

$$q_1^{\beta_1} \equiv -1 \pmod p. \tag{6}$$

From (5), we have

$$q_j^{\beta_j+1} \equiv 1 \pmod p, \text{ for } j = 2, 3, \ldots s. \tag{7}$$

Hence from (1), (6) and (7), we have

$$q_2 q_3 \ldots q_s (1 + p + \cdots + p^{2\alpha}) = q_1^{\beta_1} \prod_{j=2}^{s} q_j^{\beta_j+1} \equiv -1 \pmod p,$$

so that

$$q_2 q_3 \ldots q_s \equiv -1 \pmod p. \tag{8}$$

Now, writing $\beta = (\beta_2 + 1) \ldots (\beta_s + 1)$ we see that β is odd, since $\beta_2, \beta_3, \ldots \beta_s$ are all even.

Hence from (8), it follows that

$$q_2^\beta q_3^\beta \ldots q_s^\beta \equiv -1 \pmod p.$$

or

$$(q_2^{\beta_2+1})^{\frac{\beta}{\beta_2+1}} \ldots (q_s^{\beta_s+1})^{\frac{\beta}{\beta_s+1}} \equiv -1 \pmod p. \tag{9}$$

Hence from (7) and (9), it follows that $1 \equiv -1 \pmod p$, which shows that $p \mid 2$. This is impossible, since p is an odd prime. Hence the theorem follows.

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Near Domains of Composite Characteristic

A *near domain* is an algebraic system $(N, +, \times)$ where

(1) $(N, +)$ is a (not necessarily abelian) group with identity 0, (2) $(N, \times)$ is a semigroup, (3) $a \times (b + c) = a \times b + a \times c$ for each $a, b, c \in N$, (4) if $a \times b = 0$, then $a = 0$ or $b = 0$ (no divisors of zero), (5) at least one non-zero element is not a left identity.

Thus a near domain is a non-trivial left near-ring with no divisors of zero. Clay [1] first considered such systems (which he called near integral domains); he conjectured that like the situation in ring theory the characteristic of a finite near domain must be a prime. This note gives some simple examples to show the conjecture is false. (The conjecture was first proven to be false by Ferrero [2] in his work on planar near-rings and balanced incomplete block designs; the examples and explanation in this note are of a more elementary nature than Ferrero's.)

A key to building examples of finite near domains is the following:

Theorem If $(G, +)$ is a finite group with a fixed point-free automorphism of prime order, then it is possible to find a subgroup S of the group of automorphisms on $(G, +)$ such that

(1) each non-identity element of S is fixed point free, (2) for each non-zero $x \in G$ there corresponds some $f_x \in S$ such that by defining

$$x \times y = f_x(y) \text{ and } 0 \times y = 0, \quad \text{for each } y \in G,$$

the system $(G, +, \times)$ is a near domaine

Any abelian group of odd order has a fixed point-free automorphism of order two, e.g., $\alpha(x) = -x$, and hence will support a near domain structure. Thus there are near domains on C_9 and C_{15}.

Using a digital computer the author and Robert Whittington recently found all the neaı domains on groups of order nine and fifteen and arranged them in isomorphism classes. Up to isomorphism there are four near domains on C_9, four on $C_3 \oplus C_3$, and sixteen on C_{15}. The near domains on C_9 are easy to give and each answers in the negative Clay's conjecture since they have characteristic nine.

Example. Near domains on C_9. The only fixed point-free automorphism subgroup on C_9 is $S = \{I, \alpha\}$, where I is the identity mapping and $\alpha(x) = -x$. Taking the elements of C_9 to be 0, 1, 2, . . . , 8 in the usual fashion, each array below specifies the correspondence between elements of C_9 and elements of S and hence defines a near domain.

1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ I & I & I & I & \alpha & \alpha & \alpha & \alpha \end{pmatrix}$, 2. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ I & I & I & \alpha & I & \alpha & \alpha & \alpha \end{pmatrix}$,

3. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ I & \alpha & I & I & \alpha & \alpha & I & \alpha \end{pmatrix}$, 4. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ I & \alpha & \alpha & I & \alpha & I & I & \alpha \end{pmatrix}$.

Thus from the first array we have, for example, $5 \times 2 = f_5(2) = \alpha(2) = -2 = 7$. The multiplication tables are readily derived from the arrays.

The reader interested in further and deeper results on near domains is referred to the papers by Ligh [4] and Heatherly and Olivier [3]; the latter has a summary of all the important results on finite near domains.

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Aufgaben

Aufgabe 681. Z bezeichne die Menge der ganzen rationalen Zahlen. Eine quadratische Form

$$a x^2 - b y^2 \quad (a, b \in Z;\ a b \neq 0) \tag{*}$$

heisse linear-invariant, falls es eine lineare Transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} r & s \\ u & v \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left[r, s, u, v \in Z;\ \begin{pmatrix} r & s \\ u & v \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

so gibt, dass für alle $x, y \in Z$ gilt: $a x'^2 - b y'^2 = a x^2 - b y^2$. Man bestimme alle linear-invarianten quadratischen Formen (*). P. Wilker, Bern

Lösung (Zusammenfassung aus verschiedenen Einsendungen): Es genügt, den Fall $(a, b) = 1$, $a \geq 1$ zu betrachten. Die einzigen linear-invarianten quadratischen Formen (*) sind diejenigen, für welche die Pellsche Gleichung

$$r^2 - k^2 a b = 1$$

Lösungen $(r, k) \in Z \times Z$ mit $r^2 \neq 1$ besitzt, also $x^2 + y^2$ und $a x^2 - b y^2$, wo $b \geq 1$ und $a b$ kein Quadrat einer natürlichen Zahl ist.

Lösungen sandten J. Binz (Bern), H. Kappus (Rodersdorf SO; zwei Lösungen), I. Paasche (München, BRD) und R. Wyss (Flumenthal SO).

Anmerkung der Redaktion: H. Flanders (Tel Aviv, Israel) weist darauf hin, dass die Theoreme 58 und 74 in L. E. Dickson, Modern elementary theory of numbers (University of Chicago Press, 1939, p. 57 und 80) die obige Aussage umfassen. Der dort betrachtete Fall der Formen $a x^2 + b x y + c y^2$ wurde auch von H. Kappus behandelt.

Aufgabe 682. Let $a(1) = 1$, $a(2) = -1$. Take their negative and construct the sequence $a(1) = 1$, $a(2) = -1$, $a(3) = -1$, $a(4) = +1$. Again take the negative of this and construct the sequence,

$$a(1) = 1, \quad a(2) = -1, \quad a(3) = -1, \quad a(4) = +1, \quad a(5) = -1, \quad a(6) = +1,$$

$$a(7) = +1, \quad a(8) = -1$$

and continue the process. Determine when $a(m) = +1$ and when it is -1.

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