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# Non-Hamiltonian Square-minus-two

The square $G^2$ of a graph $G$ with vertex set $V(G)$ is defined as the graph having $V(G)$ for its vertex set, and two vertices of $G^2$ are connected by an edge in $G^2$ if and only if their distance in $G$ is at most 2. Fleishner [2] proved Nash-Williams and Plummer's Conjecture that $G^2$ is Hamiltonian for every 2-connected graph $G$; Chartrand and Kapoor [1] proved that $G^2 - v$ is Hamiltonian for every 2-connected graph $G$ with $\overline{\overline{V(G)}} \geq 4$ and every $v \in V(G)$.

It has been conjectured [4] that $G^2 - u - v$ is Hamiltonian for every 2-connected graph $G$ with $\overline{\overline{V(G)}} \geq 5$ and for every $u, v \in V(G)$.

The purpose of this note is to show that this conjecture is false (see [3]), as follows:

**Theorem 1:** For every odd integer $n$, $n \geq 3$, there exists a 2-connected graph $G = G(n)$ with $\overline{\overline{V(G)}} = 3n + 2$, such that $G^2 - u - v$ is not Hamiltonian for some $u, v \in V(G)$.

The following is even a stronger result:

**Theorem 2:** For every integer $n$, $n \geq 2$, there exists a 2-connected graph $G = G(n)$ with $\overline{\overline{V(G)}} = 8n$ and $V(G)$ contains $2n$ vertices $u_1, \ldots, u_n, v_1, \ldots, v_n$ such that $G^2 - u_i - v_i$ is not Hamiltonian for every $i$, $1 \leq i \leq n$.

We need the following simple

**Lemma:** If a graph $G$ contains a simple $u - v$ path $u, ux, x, xy, y, yz, z, zv, v$ of length 4, such that $x, y$ and $z$ are 2-valent in $G$, then every simple path in $G^2 - u - v$ that contains $y$ as an inner vertex contains also the edges $xy$ and $yz$.

*Proof:* $xy$ and $yz$ are the only edges of $G^2 - u - v$ that contain $y$ as one of their end points, hence they are contained in every simple path in $G^2 - u - v$ that contains $y$ as an inner vertex.

*Proof of Theorem 1:* For every odd $n$, $n \geq 3$, let $G(n)$ be the union of the $n$ $u - v$ paths $u, ux_i, x_i, x_iy_i, y_i, y_iz_i, z_i, z_iv, v$, for all $i = 1, \ldots, n$, where $x_i$, $y_i$ and $z_i$ are all different (see figure 1 with $n = 5$); clearly $G(n)$ is 2-connected, for all $n \geq 3$.

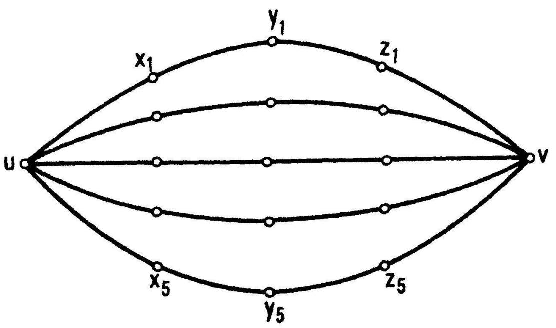


Figure 1

$G^2 - u - v$ consists of the two complete $n$-graphs $K^*$ and $K^{**}$, on the vertices $x_1, \ldots, x_n$ and $z_1, \ldots, z_n$, respectively, together with the vertices $y_1, \ldots, y_n$ and the edges $x_iy_i$, $y_iz_i$ and $x_iz_i$, for all $i = 1, \ldots, n$.

Suppose there exists a Hamiltonian cycle $h$ in $G^2 - u - v$. $h$ must contain all the vertices $y_i$, therefore it must contain by the Lemma all the edges $x_i y_i$ and $y_i z_i$, for all $i = 1, \ldots, n$; since $n > 1$, $h$ does not contain any of the egdes $x_i z_i$. $h$ therefore runs from $K^*$ to $K^{**}$ and back an odd number of times, which is impossible; hence $G^2 - u - v$ is non-Hamiltonian.

*Proof of Theorem 2:* For every integer $n \geq 2$, let the graph $G = G(n)$ consist of the $n$ cycles of length 8 of the vertices $u_i$, $x_i$, $y_i$, $z_i$, $v_i$, $z_i'$, $y_i'$, $x_i'$ and $u_i$ (in this cyclic order), for all $1 \leq i \leq n$, plus the edges $v_i u_{i+1}$ for all $1 \leq i \leq n$ (the last one of which being $v_n u_1$). $G$ is shown in Figure 2, with $n = 4$:

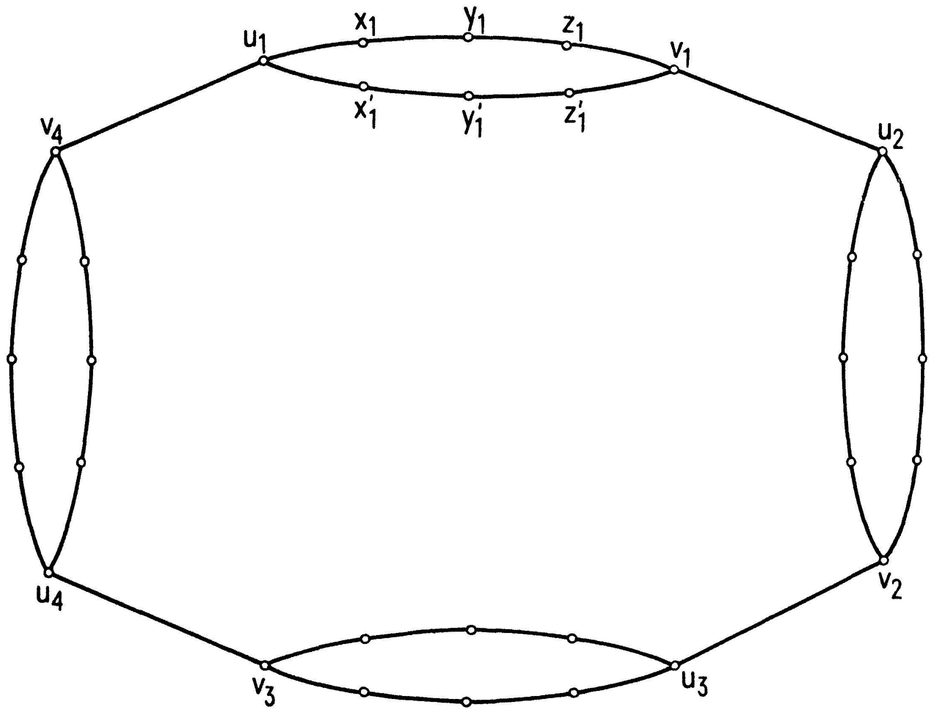


Figure 2

Clearly, $G(n)$ is a 2-connected graph for all $n \geq 2$.

Suppose that for some $i$, $1 \leq i \leq n$, $G^2 - u_i - v_i$ contains a simple cycle $c$ that contains both $y_i$ and $y_i'$; $c$ contains, by the Lemma, all the four edges $x_i y_i$, $y_i z_i$, $x_i' y_i'$ and $y_i' z_i'$. $c$ does not contain the edge $x_i z_i$ ($x_i' z_i'$), since otherwise $c$ would have at least two connected components, one of which being a cycle of length 3. The vertex $x_i$, as any vertex of $c$, is of valence 2 in $c$, hence either the edge $x_i v_{i-1}$ or else the edge $x_i x_i'$ is in $c$; similarily for the vertices $x_i'$, $z_i$ and $z_i'$. It follows that $c$ contains either the edge $x_i x_i'$ or else the two edges $x_i v_{i-1}$ and $x_i' v_{i-1}$; similarly, $c$ contains either the edge $z_i z_i'$ or else the two edges $z_i u_{i+1}$ and $z_i' u_{i+1}$. As a result, $c$ is of length at most 8, while $G^2 - u_i - v_i$ has $8n - 2$ vertices, $n \geq 2$; since $8 < 8n - 2$, for all $n \geq 2$, it follows that $c$ does not contain all the vertices of $G^2 - u_i - v_i$, therefore $G^2 - u_i - v_i$ is non-Hamiltonian. This completes the proof of Theorem 2.

*Remark:* It follows immediately from [1] that if $G$ is a 2-connected graph, then $G^2 - u - v$ has a Hamiltonian path for every $u, v \in V(G)$.

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REFERENCES

[1] G. CHARTRAND and S. F. KAPOOR, *The Square of Every 2-Connected Graph is 1-Hamiltonian,* to appear.
[2] H. FLEISHNER, *The Square of Every Nonseparable Graph is Hamiltonian,* to appear.
[3] J. ZAKS, announcement 1(B), Graph Theory Newsletter Vol. 1 (No. 4), 1972 edited by S. F. Kapoor, W.M.U. Kalamazoo, Michigan), p. 7.
[4] (anonymous) Problem 1, Graph Theory Newsletter, Vol. 1. (No. 2), (1971), p. 3.

# Kleine Mitteilungen

## Proof of a Conjecture of H. Hadwiger

As part of a research problem [2], Hadwiger conjectured that every simple closed curve in $E^3$ admits a nontrivial inscribed parallelogram. Schnirelman's method [4] [1] leads immediately to the following result:

**Theorem:** *Every simple closed* $C^2$ *curve in* $E^3$ *admits a nontrivial inscribed rhombus.*

Outline of proof: The statement for plane curves has been proved by Schnirelman [4] [1]. Every simple closed curve in $E^3$ is homotopic to a plane Jordan curve. If the curve in $E^3$ is not knotted, the homotopy is in fact an isotopy. If the curve is a knot, it may be deformed into a plane Jordan curve through a $C^2$-homotopy $F(\alpha, t)$, $0 \le \alpha \le 2\pi$, $0 \le t \le 1$, for which $F(\alpha, t_0)$ is a simple closed curve except for finitely many values $t_0$ for which $F(\alpha, t_0)$, $0 \le \alpha \le 2\pi$, is a curve with one simple transversal selfintersection. Because of the compactness of the sets involved, a given smooth homotopy can be locally modified to satisfy the given conditions. The parametrization can be chosen so that the Jacobian matrix of $F$ is nowhere singular. The theorem will be proved if we can show that it holds for all curves $F(\alpha, t)$, $t_0 \le t < t_0 + \varepsilon$ if it holds for $F(\alpha, t_0)$.

By hypothesis, there exist four distinct parameter values $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ so that for $F_i = F(\alpha_i, t_0)$ we have

$$|F_1 - F_2| = |F_2 - F_3| = |F_3 - F_4| = |F_4 - F_1| \; (\neq 0) \qquad (1)$$
$$\det\,(F_1 - F_2,\; F_1 - F_3,\; F_1 - F_4) = 0$$

where det denotes the determinant. The problem is to find four points $F_i^*$ on $F(\alpha, t)$, $t_0 \le t \le t_0 + \varepsilon$, that also satisfy conditions (1). We develop in a Taylor polynomial,

$$F_i^* = F_i + \frac{\delta F_i}{\delta x_i}\, \Delta\alpha_i + \frac{\delta F_i}{\delta t}\, \Delta t + o(\Delta\alpha_i, \Delta t)$$

introduce the expression in (1) and develop as well. An appropriate form of the inverse function theorem says that under our differentiability assumptions the $\Delta\alpha_i$ can be found if the linearized problem obtained by putting all $o(\Delta\alpha_i, \Delta t) = 0$, can be solved. From (1) one obtains a system of four nonhomogeneous linear equations (that can immediately be written down) for the four unknowns $\Delta\alpha_i$ $(i = 1, 2, 3, 4)$. The matrix of the system has the form