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A Criterion for *n*-Fold Transitivity of Transformation Groups

Let G be a group and let X be a nonempty set. An *action* * on X is a function *: $G \times X \rightarrow X$ such that for every g, $h \in G$ and $x \in X$, (i) (gh) *x = g * (h * x) and (ii) 1 * x = x.

A triple (G, X, *) where * is an action of G on X is called a *transformation group*. For $S \subseteq X$ the stability subgroup of S is $G_S = \{g \in G \mid g * s = s \text{ for every } s \in S\}$. (We will write G_x instead of $G_{(x)}$.)

If *n* is a positive integer, we say that *G* is *n*-fold transitive whenever for every two sequences x_1, x_2, \ldots, x_n and y_1, y_2, \ldots, y_n each consisting of *n* distinct elements of *X*, there exists $g \in G$ such that $g * x_i = y_i$ for every $i = 1, 2, \ldots, n$.

We note that if * is an action of G on X, then for any $S \subseteq X$, * induces an action of G_S on X - S.

The next theorem is well known (see, for example, [1], Theorem 9.1).

Theorem 1: Let (G, X, *) be transitive. Then for $n \ge 2$, (G, X, *) is *n*-fold transitive iff there exists an $x \in X$ such that $(G_x, X - \{x\}, *)$ is (n-1)-fold transitive.

It is our purpose in this note to derive a corollary (Theorem 2) of this theorem which is sometimes more convenient to use. The essential idea is to replace the transitive condition on (G, X, *) by a restriction on the stability subgroups.

Lemma 1: If (G, X, *) is a transformation group, then (G, X, *) is 2-fold transitive iff there exists an $x \in X$ such that $G_x \neq G$ and $(G_x, X - \{x\}, *)$ is transitive.

Proof: Clearly if (G, X, *) is 2-fold transitive then the given condition holds for any $x \in X$.

Now suppose $x \in X$ such that $G_x \neq G$ and $(G_x, X - \{x\}, *)$ is transitive. Let $y, z \in X$. If $y, z \in X - \{x\}$, then there exists $g \in G_x$ such that g * y = z. If y = z = x, then 1 * y = z. If y = x and $z \neq x$, then since $G_x \neq G$, there exists $h \in G$ such that $h * x \neq x$. So there is an $r \in G_x$ such that r * (h * x) = z and so (rh) * x = z. If y = x, z = x and h is as before, then there exists $t \in G_x$ such that t * y = h * x = h * z so that $(h^{-1}t) * y = z$. Hence (G, X, *) is transitive so that by Theorem 1 it is 2-fold transitive.

Lemma 2: Let $n \ge 2$ and |X| > 1. Then (G, X, *) is *n*-fold transitive iff there exists an $x \in X$ such that $G_x \neq G$ and $(G_x, X - \{x\}, *)$ is (n - 1)-fold transitive.

Proof: Assume $G_x \neq G$ and $(G_x, X - \{x\}, *)$ is (n - 1)-fold transitive. Then by Lemma 1, (G, X, *) is transitive and hence by Theorem 1 it is *n*-fold transitive. If (G, X, *) is *n*-fold transitive, then the given condition holds for all $x \in X$.

Theorem 2: For $|X| \ge n \ge 2$, (G, X, *) is *n*-fold transitive iff there exists $S \subseteq X$ with $S = \{t_1, t_2, \ldots, t_{n-1}\}$ such that if $S_k = \{t_1, t_2, \ldots, t_k\}$ for each $k = 1, 2, \ldots, n-1$, then

a) $G_{t_1} \neq G$ and $G_{S_k} \neq G_{S_{k+1}}$ for all k = 1, 2, ..., n-1; and

b) $(G_S, X - S, *)$ is transitive.

Proof: Since any n-fold transformation group clearly satisfies (a) and (b), we need only show the other half.

The case n = 2 is the content of Lemma 1.

Suppose the theorem holds for all integers greater than one and less than n. Let $S \subseteq X$ be $S = \{t_1, t_2, \ldots, t_{n-1}\}$ such that Conditions (a) and (b) hold. Then $S^* = \{t_2, \ldots, t_{n-1}\}$ satisfies the conditions of the theorem for the transformation group $(G_{t_1}, X - \{t_1\}, *)$ and hence this transformation group is (n - 1)-fold transitive. But then by Lemma 2, (G, X, *) is *n*-fold transitive.

We next consider an application of this result. Let k be a field and let G be the group GL(k, 2) of all nonsingular 2×2 matrices over k. Let * be the action of G on $k \cup \{\infty\}$ defined by

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} * z = \begin{cases} \frac{\alpha z + \beta}{\gamma z + \delta}, & \text{if } z \neq \infty, \ \gamma z + \delta \neq 0 \\ \infty, & \text{if } z \neq \infty, \ \gamma z + \delta = 0 \\ \alpha / \gamma, & \text{if } z = \infty, \ \gamma \neq 0 \\ \infty, & \text{if } z = \infty, \ \gamma = 0. \end{cases}$$

We will apply the previous result to show that (G, X, *) is 3-fold transitive. First we note the following special case of Theorem 2 obtained by letting n = 3.

Theorem 3: For $|X| \ge 3$, (G, X, *) is 3-fold transitive iff there exist $x, y \in X$ such that $G_x \neq G$, $G_{\{x,y\}} \neq G_x$ and $(G_{\{x,y\}}, X - \{x, y\}, *)$ is transitive.

Note that $G_{\{x,y\}} = G_x \cap G_y$. It is easy to see that

$$G_{\infty} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| a, b, c \in k, ac \neq 0 \right\}$$

and

$$G_{\mathbf{0}} = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \middle| a, b, c \in k, ac \neq 0 \right\}.$$

So

$$G_{\{0,\infty\}} = G_0 \cap G_{\infty} = \left\{ \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \middle| a, c \in k, ac \neq 0 \right\}.$$

Hence $G_0 \neq G$, $G_{\{0,\infty\}} \neq G_0$. It is also clear that $(G_{\{0,\infty\}}, k-\{0\}, *)$ is transitive, for if $x \neq 0$ and $y \neq 0$, then

$$\begin{bmatrix} y & 0 \\ 0 & x \end{bmatrix} * x = y \, .$$

Hence by Theorem 3, (G, X, *) is 3-fold transitive. We note that (G, X, *) is not 4-fold transitive, for then $(G_{\{0,\infty\}}, k - \{0\}, *)$ would be 2-fold transitive.

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On 1-Factorability and Edge-Colorability of Cartesian Products of Graphs

There is no characterization of 1-factorable graphs. Thus, it is natural that many of the results on this topic have been the determination of classes of 1-factorable graphs. The object of this paper is to present a sufficient condition for the 1-factorability of the cartesian product of two graphs. We begin with some notation and definitions.

The vertex set of a graph G will be denoted by V(G) and its edge set by E(G). In this paper we consider only finite, undirected graphs without loops or multiple edges. Let G and H be two nonempty graphs for which V(G) = V(H) and $E(G) \cap E(H) = \Phi$; then the graph G' is the sum of G and H, written G' = G + H, if V(G') = V(G) and $E(G') = E(G) \cup E(H)$. A 1-factor of a graph G is a spanning 1-regular subgraph of G. A graph is 1-factorable if it can be expressed as a sum of edge-disjoint 1-factors. The cartesian product (or product) of the graph G with the graph H, denoted by $G \times H$, is defined by: $V(G \times H) = V(G) \times V(H)$; $E(G \times H) = \{[(u_1, v_1), (u_2, v_2)] | u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)\}$.

An assignment of *n* colors to the edges of a nonempty graph *G* so that adjacent edges are colored differently is an *n*-edge-coloring of *G*. The minimum *n* for which a graph *G* is *n*-edge-colorable is its edge-chromatic number $\chi_1(G)$. By a theorem of Vizing [2], the edge-chromatic number $\chi_1(G)$ of a graph *G* is bounded by: $\Delta(G) \leq \chi_1(G) \leq$ $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of *G*. If *G* is regular, then *G* is 1-factorable if and only if $\chi_1(G) = \Delta(G)$. Hence any theorem concerning the 1-factorability of regular graphs has as an immediate corollary a result concerning edge-colorability, which is useful since there is also no characterization of those graphs which are $\Delta(G)$ -edge-colorable. For other notations and definitions, we follow [1].

If K_2 denotes the complete graph on two vertices, then $K_2 \times H$, where H is any regular graph, is shown to be 1-factorable in the following lemma.