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Hence  $G_0 \neq G$ ,  $G_{\{0,\infty\}} \neq G_0$ . It is also clear that  $(G_{\{0,\infty\}}, k - \{0\}, *)$  is transitive, for if  $x \neq 0$  and  $y \neq 0$ , then

$$\begin{bmatrix} y & 0 \\ 0 & x \end{bmatrix} * x = y.$$

Hence by Theorem 3,  $(G, X, *)$  is 3-fold transitive. We note that  $(G, X, *)$  is not 4-fold transitive, for then  $(G_{\{0,\infty\}}, k - \{0\}, *)$  would be 2-fold transitive.

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### REFERENCE

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## On 1-Factorability and Edge-Colorability of Cartesian Products of Graphs

There is no characterization of 1-factorable graphs. Thus, it is natural that many of the results on this topic have been the determination of classes of 1-factorable graphs. The object of this paper is to present a sufficient condition for the 1-factorability of the cartesian product of two graphs. We begin with some notation and definitions.

The vertex set of a graph  $G$  will be denoted by  $V(G)$  and its edge set by  $E(G)$ . In this paper we consider only finite, undirected graphs without loops or multiple edges. Let  $G$  and  $H$  be two nonempty graphs for which  $V(G) = V(H)$  and  $E(G) \cap E(H) = \emptyset$ ; then the graph  $G'$  is the *sum* of  $G$  and  $H$ , written  $G' = G + H$ , if  $V(G') = V(G)$  and  $E(G') = E(G) \cup E(H)$ . A *1-factor* of a graph  $G$  is a spanning 1-regular subgraph of  $G$ . A graph is *1-factorable* if it can be expressed as a sum of edge-disjoint 1-factors. The *cartesian product* (or *product*) of the graph  $G$  with the graph  $H$ , denoted by  $G \times H$ , is defined by:  $V(G \times H) = V(G) \times V(H)$ ;  $E(G \times H) = \{[(u_1, v_1), (u_2, v_2)] \mid u_1 = u_2 \text{ and } v_1 v_2 \in E(H), \text{ or } v_1 = v_2 \text{ and } u_1 u_2 \in E(G)\}$ .

An assignment of  $n$  colors to the edges of a nonempty graph  $G$  so that adjacent edges are colored differently is an  *$n$ -edge-coloring* of  $G$ . The minimum  $n$  for which a graph  $G$  is  $n$ -edge-colorable is its *edge-chromatic number*  $\chi_1(G)$ . By a theorem of Vizing [2], the edge-chromatic number  $\chi_1(G)$  of a graph  $G$  is bounded by:  $\Delta(G) \leq \chi_1(G) \leq \Delta(G) + 1$ , where  $\Delta(G)$  is the maximum degree of  $G$ . If  $G$  is regular, then  $G$  is 1-factorable if and only if  $\chi_1(G) = \Delta(G)$ . Hence any theorem concerning the 1-factorability of regular graphs has as an immediate corollary a result concerning edge-colorability, which is useful since there is also no characterization of those graphs which are  $\Delta(G)$ -edge-colorable. For other notations and definitions, we follow [1].

If  $K_2$  denotes the complete graph on two vertices, then  $K_2 \times H$ , where  $H$  is any regular graph, is shown to be 1-factorable in the following lemma.

**Lemma:** If  $H$  is a regular graph, then  $K_2 \times H$  is 1-factorable.

*Proof.* If  $H$  is 1-factorable, then the result follows immediately. Hence we consider the case that  $H$  is not 1-factorable. If  $H$  is an  $r$ -regular graph, then by a previous remark,  $\chi_1(H) = r+1$ . Let an  $(r+1)$ -edge-coloring of  $H$  be given and let  $C_1, C_2, \dots, C_{r+1}$  be the edge-color classes of  $E(H)$ . Now  $K_2 \times H$  contains two disjoint copies of  $H$ . Let the  $(r+1)$ -edge-coloring of  $H$  be applied to these disjoint copies, and assign to each edge  $[(u_1, v), (u_2, v)]$  of  $K_2 \times H$  the only color among the  $r+1$  colors which was assigned to no edge of  $H$  incident with  $v$ . Hence  $K_2 \times H$  may be  $(r+1)$ -edge-colored. But  $K_2 \times H$  is  $(r+1)$ -regular. Hence  $\chi_1(K_2 \times H) = r+1$ , and  $K_2 \times H$  is 1-factorable.

We now state and prove the main result.

**Theorem:** If  $G$  is a 1-factorable graph and  $H$  is a regular graph, then  $G \times H$  is a 1-factorable graph.

*Proof:* Let  $G$  be a 1-factorable,  $r$ -regular graph of order  $p_1$  with 1-factors  $G_1, G_2, \dots, G_r$ , and let  $H$  be an  $s$ -regular graph of order  $p_2$ . Then consider the subgraphs  $G_1 \times H, G_2 \times \bar{K}_{p_2}, \dots, G_r \times \bar{K}_{p_2}$  of  $G \times H$ , where  $\bar{K}_{p_2}$  denotes the graph consisting of  $p_2$  isolated vertices. Note that these subgraphs are mutually edge-disjoint subgraphs spanning  $G \times H$ , and  $G \times H = G_1 \times H + \sum_{i=2}^r G_i \times \bar{K}_{p_2}$ . Moreover, the subgraphs  $G_2 \times \bar{K}_{p_2}, \dots, G_r \times \bar{K}_{p_2}$  are 1-regular and thus are 1-factors of  $G \times H$ . Hence if  $G_1 \times H$  is 1-factorable,  $G \times H$  is 1-factorable. Now  $G_1 \times H$  is a spanning  $(s+1)$ -regular subgraph of  $G \times H$  consisting of  $p_1/2$  components each of which is isomorphic to  $K_2 \times H$ . By the Lemma,  $K_2 \times H$  is 1-factorable and of regularity  $s+1$ . Let the 1-factors of  $K_2 \times H$  be  $F_1, F_2, \dots, F_{s+1}$  in a 1-factorization of  $K_2 \times H$ . Select in every component of  $G_1 \times H$ , the same 1-factor  $F_k$ , where  $1 \leq k \leq s+1$ , and designate the resultant subgraph of  $G_1 \times H$  by  $F'_k$ . Then by the choice of  $F'_k$  it follows that  $F'_k$  is a spanning 1-regular subgraph of  $G_1 \times H$ , and hence a 1-factor of  $G_1 \times H$ . In a like manner mutually edge-disjoint 1-factors  $F'_1, F'_2, \dots, F'_{s+1}$  of  $G_1 \times H$  can be obtained from each of  $F_1, F_2, \dots, F_{s+1}$ , respectively. Therefore  $G_1 \times H$  is 1-factorable, which implies that  $G \times H$  is also 1-factorable as previously indicated.

*Corollary:* If  $G$  and  $H$  are regular graphs, and  $\chi_1(G) = \Delta(G)$ , then  $\chi_1(G \times H) = \Delta(G) + \Delta(H)$ .

We remark that the theorem gives a sufficient condition for 1-factorability which is, however, not a necessary condition, since 1-factorable products of two non-1-factorable graphs are known. An example of this is the cartesian product of the Petersen graph with a triangle.

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