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Autor: Goldberg, M.
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New Rectifiable Tetrahedra

1. Introduction. Hilbert's third problem [1] asked if there were a pair of polyhedra of equal volume such that one polyhedron could not be divided into a finite number of pieces to make the other. Bricard [2] and Dehn [3] showed that the regular tetrahedron and the cube are such a pair. There still remained unanswered the problem of describing and enumerating those other tetrahedra which are equivalent to cubes by dissection. Such tetrahedra are here called rectifiable tetrahedra.

The methods of obtaining rectifiable tetrahedra consist of dividing rectifiable prisms into congruent tetrahedra, and then adding or subtracting rectifiable prisms and rectifiable tetrahedra with the hope of discovering new shapes of rectifiable tetrahedra. Several investigators have used these methods and found examples. Their findings are summarized in a paper by the author [4]. Since completeness has never been demonstrated, there is still room for further investigation. The following note describes several newly-found rectifiable tetrahedra by the same methods.

2. Dissection theorems

Definitions. If a polyhedron can be cut into a finite number of pieces to form another polyhedron, then the two polyhedra are said to be *equidecomposable.* If one of the polyhedra is a cube, then the other is said to be *rectifiable.*

Theorem 1. (Gerling 1844, Bricard 1896) Two isometric tetrahedra (mirror images of each other) are equidecomposable.

Proof: From the center F of the circumscribed sphere of the tetrahedron $ABCD$, drop a perpendicular FE to the face ABC. Then the planes EFA, EFB, EFC, ABF, BCF, CAF cut off the tetrahedra numbered 1, 2, 3 adjoining the face ABC. Similarly, three tetrahedra are obtained for each of the other faces, making a total of 12 tetrahedra into which $ABCD$ is divided. The three tetrahedra 1, 2, 3 can be assembled in the reverse order, as shown in Figure 1, to form the face of the symmetric tetrahedron $A'B'C'D'$. The other faces are similarly treated.

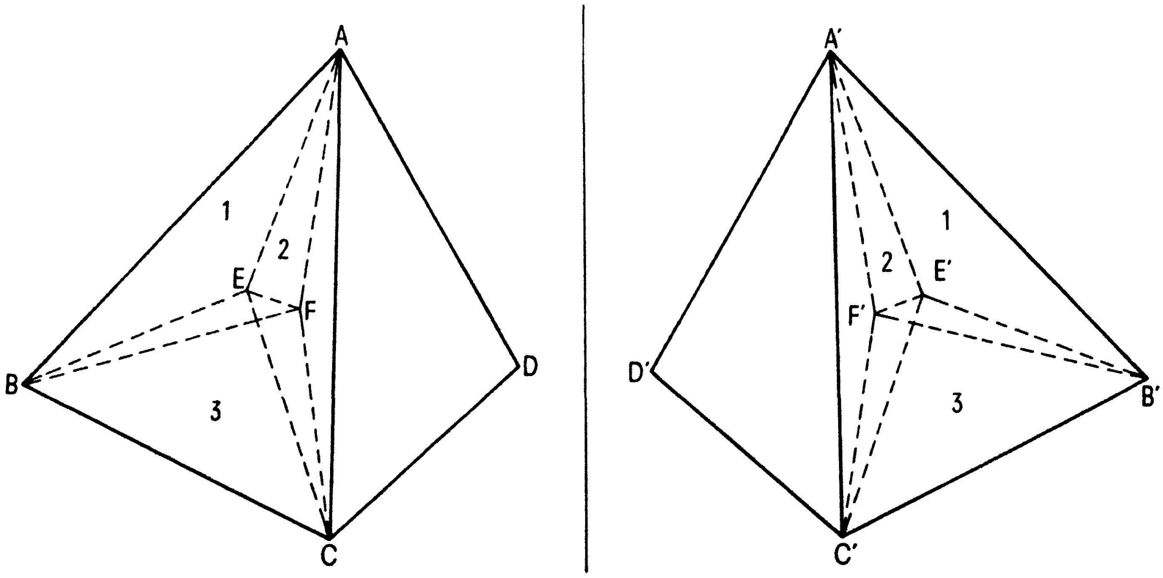

Fig. 1. Isometric tetrahedra are equidecomposable

If the center F of the circumscribed sphere of the tetrahedron $ABCD$ falls outside of the tetrahedron, then the tetrahedron is made of a combination of additions and subtractions of the 12 component tetrahedra.

It is well known that all prisms are rectifiable. Therefore, if a prism is divisible into congruent tetrahedra, then these tetrahedra are also rectifiable. Sydler [5] showed that a polyhedron, which is the sum or difference of two rectifiable polyhedra, is also rectifiable. Also, if Theorem 1 is used, we can state another theorem of Sydler as follows.

Theorem 2 (Sydler 1943). If n given similar tetrahedra (directly similar or mirror images) can be dissected and then re-assembled into a rectifiable polyhedron, then each of these tetrahedra is rectifiable.

3. Derivation of the special rectifiable tetrahedra

If a corner of a cube is cut by a plane whose intercepts on the edges are proportional to τ, $1, 1/\tau$, where $\tau = (1 + \sqrt{5})/2$, then the tetrahedron that is cut off by the plane is Sydler's tetrahedron T_1. By assembling 120 of these tetrahedra into a polyhedron of 30 faces and dissecting it into prisms, Sydler showed that T_1 is rectifiable. A simpler demonstration is shown in the following theorem.

Theorem 3. The Sydler tetrahedron T_1 is rectifiable.

Proof: The Hill tetrahedron $H_1(\alpha)$ is rectifiable since three of them make a prism. Figure 2 shows an $H_1(\alpha)$ tetrahedron for $\alpha = 2\pi/5$. It is divided into four similar tetrahedra. Three of these pieces are the same size and are designated by T_1. The smaller similar piece is designated by t_1. Hence, by Theorem 2, the tetrahedron T_1 is rectifiable.

There are many relations connecting the various rectifiable tetrahedra. In attempting to find new shapes by addition and subtraction, one frequently finds a shape already known. Table 1 indicates the character of the tetrahedral parts obtained by dividing a rectifiable tetrahedron into two pieces by a plane through an edge. The designated angle is the dihedral angle at the edge. Where only one angle is indicated, the dihedral angle is bisected equally. In other cases, the unequal parts into which the

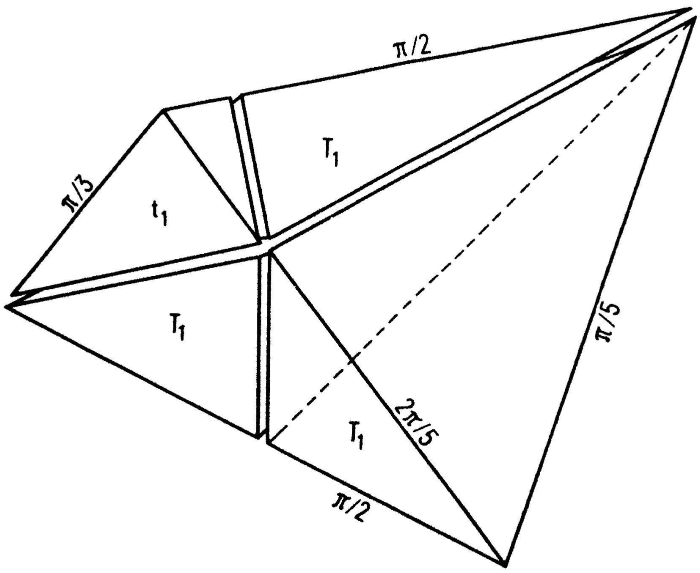

Fig. 2. Rectifiable tetrahedron $H_1(2\pi/5)$ divided into four tetrahedra of Type T_1

Table 1

TETRA.	ANGLE	PARTS
$H_1(\alpha)$	$\pi - 2\alpha$	$H_2(\alpha)+H_2(\alpha)$
$H_1(\alpha)$	$\pi/3$	$H_3(\alpha)+H_3(\alpha)$
$H_1(2\pi/5)$		$3T_1+t_1$
$H_1(2\pi/5)$	$2\pi/5$	T_2+T_3
$H_2(\pi/5)$	$\pi/5+\pi/10$	T_6+T_{20}
T_1	$2\pi/3$	T_1+T_1
T_2	$2\pi/5$	$T_{16}+T_{18}$
T_2	$2\pi/5$	T_1+T_1
T_3	$\pi/5+2\pi/5$	T_4+T_5
T_3	$\pi/10+3\pi/10$	T_9+T_{14}
T_3	$2\pi/5$	T_7+T_{16}
T_4	$2\pi/3$	T_3+T_{13}
T_4	$2\pi/5+\pi/5$	$T_{17}+T_{18}$
T_5	$2\pi/5$	T_1+T_1
T_5	$2\pi/5$	$H_1(2\pi/5)+T_{21}$
T_6	$4\pi/5$	T_1+T_1
T_6	$\pi/5+3\pi/5$	T_4+T_3
T_7	$\pi/3$	T_8+T_8
T_7	$\pi/5$	T_9+T_9
T_7	$3\pi/5$	$T_{10}+T_{10}$
T_7	$\pi/3, \pi/5, 3\pi/5$	$4T_{11}$
T_7	$\pi/5+2\pi/5$	T_4+T_{21}
T_5	$\pi/10+3\pi/10$	$H_2(2\pi/5)+T_{22}$
T_5	$3\pi/10+\pi/10$	$T_{24}+H_2(2\pi/5)$
T_{13}	$2\pi/5+\pi/5$	T_3+T_{16}

TETRA.	ANGLE	PARTS
T_{10}	$\pi/5+\pi/10$	T_4+T_{15}
T_{11}	$3\pi/5$	$T_{14}+T_{14}$
T_{13}	$\pi/5$	$T_{15}+T_{15}$
T_{14}	$\pi/10+\pi/5$	T_9+T_{16}
T_{16}	$2\pi/3$	$T_{17}+T_{21}$
T_{16}	$2\pi/5$	T_6+T_{18}
T_{17}	$2\pi/5$	$H_1(\pi/5)+T_{16}$
T_{17}	$2\pi/3$	T_3+T_{23}
T_{18}	$\pi/3$	$T_{19}+T_{19}$
T_{18}	$\pi/5$	$T_{20}+T_{20}$
T_{18}	$2\pi/3$	T_4+T_{17}
T_{21}	$2\pi/5$	T_4+T_{13}
T_{22}	$\pi/10+\pi/5$	$H_2(2\pi/5)+T_{21}$
T_{23}	$3\pi/5$	$T_{24}+T_{24}$
T_{23}	$\pi/3$	$T_{25}+T_{25}$
T_{23}	$2\pi/5$	T_3+T_{17}
T_{21}	$\pi/5+2\pi/5$	T_2+T_{23}
T_{23}	$2\pi/5+\pi/5$	T_5+T_{21}
T_{24}	$\pi/5+\pi/10$	$T_{21}+H_2(2\pi/5)$
T_{21}	$3\pi/10+\pi/10$	$T_{10}+T_{15}$
T_4	$\pi/5+2\pi/5$	$H_1(\pi/5)+T_2$
$H_1(\pi/5)$	$\pi/5+2\pi/5$	T_6+T_{16}
T_{26}		$(T_{16}+T_{16})/2$
T_{27}		$(T_{21}+T_{21})/2$
T_{26}	$(\pi/3-\alpha_{13})+\pi/3$	$T_{27}+T_7$

dihedral angle is divided is indicated by two angles. For one of the divisions of T_7, all the angles are bisected to divide T_7 into four parts, each of which is T_{11}.

If one of the parts is known to be rectifiable, then the other part is also rectifiable. The new tetrahedra are numbered sequentially as found. Table 2 gives the dihedral angles and the relative lengths of the edges of the newly-found rectifiable tetrahedra. There is no claim for completeness; there may be many more to be discovered. Sydler [6] showed that Dehn's conditions are necessary and sufficient for a tetrahedron to be rectifiable. These conditions have not been employed to find the listed tetrahedra, although they are compatible. A guide to satisfactory subdivisions is the selection of pairs of known rectifiable tetrahedra which have a trihedral angle in common and an included face in common. Then the smaller tetrahedron can be subtracted from the larger to obtain another rectifiable tetrahedron.

The procedure that has been used is clarified by considering several examples. The bisecting plane ABE of angle $2\pi/3$ of the rectifiable tetrahedron, designated by

Table 2

EDGE	T_{13}		T_{14}		T_{15}		T_{16}		T_{17}	
	LENGTH	ANGLE	LENGTH	ANGLE	LENGTH	ANGLE	LENGTH	ANGLE	LENGTH	ANGLE
AB	$\sqrt{10+2\sqrt{5}}$	$\pi/5$	$\sqrt{10+2\sqrt{5}}/2$	$\pi/5$	$\sqrt{10+2\sqrt{5}}$	$\pi/10$	$\sqrt{10-2\sqrt{5}}$	$\pi/5$	$\sqrt{10+2\sqrt{5}}$	$\pi/5$
AC	$\sqrt{18-6\sqrt{5}}$	$\pi/3$	$\sqrt{18-6\sqrt{5}}$	$\pi/3$	$\sqrt{18-6\sqrt{5}}$	$\pi/3$	$2\sqrt{3}$	$\pi/3$	$\sqrt{3}(1+\sqrt{5})$	$\pi/3$
AD	$2\sqrt{5}-2$	$\pi/2$	$2\sqrt{5}-2$	$\pi/2$	$\sqrt{16-5\sqrt{5}}$	$\pi-\alpha_3$	4	$\pi/2$	4	$\pi/2$
BC	$2\sqrt{5}-2$	$\pi/2$	$\sqrt{30-10\sqrt{5}}/2$	$\pi-\alpha_2$	$2\sqrt{5}-2$	$\pi/2$	$\sqrt{18-6\sqrt{5}}$	$2\pi/3$	$2\sqrt{3}$	$\pi/3$
BD	$\sqrt{18-6\sqrt{5}}$	$\pi/3$	$\sqrt{30-10\sqrt{5}}/2$	α_2	$\sqrt{16-5\sqrt{5}}$	α_3	$\sqrt{10-2\sqrt{5}}$	$2\pi/5$	$\sqrt{18-6\sqrt{5}}$	$2\pi/3$
CD	$\sqrt{50-22\sqrt{5}}$	$3\pi/5$	$\sqrt{50-22\sqrt{5}}$	$3\pi/10$	$\sqrt{5-2\sqrt{5}}$	$3\pi/5$	$\sqrt{20-8\sqrt{5}}$	$\pi/5$	$\sqrt{10-2\sqrt{5}}$	$2\pi/5$

EDGE	T_{18}		T_{19}		T_{20}		T_{21}		T_{22}	
	LENGTH	ANGLE	LENGTH	ANGLE	LENGTH	ANGLE	LENGTH	ANGLE	LENGTH	ANGLE
AB	$\sqrt{20-8\sqrt{5}}$	$\pi/5$	$\sqrt{3}(\sqrt{5}-1)$	$\pi/6$	$\sqrt{10+2\sqrt{5}}$	$\pi/10$	$\sqrt{10-2\sqrt{5}}$	$\pi/5$	$\sqrt{3}(1+\sqrt{5})/2$	$\pi/3$
AC	$\sqrt{10-2\sqrt{5}}$	$\pi/5$	$\sqrt{10-2\sqrt{5}}$	$\pi/5$	$\sqrt{10-2\sqrt{5}}$	$\pi/5$	$2\sqrt{3}$	$\pi/3$	$2\sqrt{3}$	$\pi/3$
AD	$\sqrt{18-6\sqrt{5}}$	$2\pi/3$	$\sqrt{34-6\sqrt{5}}/2$	$\pi-\alpha_8$	$\sqrt{26+2\sqrt{5}}/2$	$\pi-\alpha_9$	$\sqrt{10-2\sqrt{5}}$	$3\pi/5$	$\sqrt{10+2\sqrt{5}}$	$2\pi/5$
BC	$\sqrt{18-6\sqrt{5}}$	$2\pi/3$	$2\sqrt{3}$	$2\pi/3$	$2\sqrt{3}$	$2\pi/3$	$\sqrt{10+2\sqrt{5}}$	$2\pi/5$	$\sqrt{26-2\sqrt{5}}/2$	$\pi-\alpha_{10}$
BD	$\sqrt{3}(\sqrt{5}-1)$	$\pi/3$	$\sqrt{34-6\sqrt{5}}/2$	α_8	$\sqrt{26+2\sqrt{5}}/2$	α_9	$2\sqrt{5}-2$	$\pi/2$	$\sqrt{26-2\sqrt{5}}/2$	α_{10}
CD	$\sqrt{20-8\sqrt{5}}$	$\pi/5$	$\sqrt{10+2\sqrt{5}}/2$	$\pi/5$	$\sqrt{3}(\sqrt{5}-1)/2$	$\pi/3$	$\sqrt{3}(\sqrt{5}-1)$	$\pi/3$	$\sqrt{10-2\sqrt{5}}$	$3\pi/10$

EDGE	T_{23}		T_{24}		T_{25}		T_{26}			
	LENGTH	ANGLE	LENGTH	ANGLE	LENGTH	ANGLE	LENGTH	ANGLE		
AB	$2\sqrt{3}$	$\pi/3$	$2\sqrt{3}$	$\pi/6$	$\sqrt{18-6\sqrt{5}}$	α_{12}	$2\sqrt{3}$	$\pi/3-\alpha_{12}$		
AC	$\sqrt{18-6\sqrt{5}}$	$\pi/3$	$\sqrt{18-6\sqrt{5}}$	$\pi/3$	$2\sqrt{3}$	$\pi/3$	$2\sqrt{3}$	α_{12}		
AD	$\sqrt{10-2\sqrt{5}}$	$2\pi/5$	$\sqrt{9-2\sqrt{5}}$	$\pi-\alpha_{11}$	$4\sqrt{3-\sqrt{5}}$	$\pi/2$	$\sqrt{10-2\sqrt{5}}$	$4\pi/5$		
BC	$\sqrt{10-2\sqrt{5}}$	$2\pi/5$	$\sqrt{10-2\sqrt{5}}$	$2\pi/5$	$\sqrt{10-2\sqrt{5}}$	$3\pi/5$	$4\sqrt{2}$	$\pi/2$		
BD	$\sqrt{18-6\sqrt{5}}$	$\pi/3$	$\sqrt{9-2\sqrt{5}}$	α_{11}	$\sqrt{18-6\sqrt{5}}$	$2\pi/3-\alpha_{12}$	$\sqrt{10+2\sqrt{5}}$	$2\pi/5$		
CD	$\sqrt{20-8\sqrt{5}}$	$3\pi/5$	$\sqrt{5-2\sqrt{5}}$	$3\pi/5$	$\sqrt{20-8\sqrt{5}}$	$\pi/5$	$\sqrt{3}(\sqrt{5}-1)$	$\pi/3$		

T_4, divides the tetrahedron into two unequal tetrahedra, as shown in Figure 3. The smaller of these is the same as T_3. The larger tetrahedron of these two has a new shape and is designated by T_{13}. The tetrahedron T_{13} of paper [4] was cancelled because it is a special case of Hill's third type with $\alpha = \pi/4$.

The new tetrahedron T_{13} has an axis of symmetry which passes through the midpoints of opposite edges. A plane, which bisects the dihedral angle at the short edge, divides T_{13} into two congruent tetrahedra designated by T_{14}. A plane, which bisects the dihedral angle at the long edge, divides T_{13} into two congruent tetrahedra designated by T_{15}. The foregoing relations can be described by the notations $T_{13} = T_4 - T_3$, $T_{14} = 1/2\, T_{13}$ (short), $T_{15} = 1/2\, T_{13}$ (long).

Tetrahedra T_2, T_3, T_4, T_5 and T_6 are obtained by appropriate combinations of tetrahedra of type T_1. However, Lenhard obtained T_7 by a new process, namely, by subtracting four tetrahedra of type T_1 from a rectangular prism of edges τ, $1, 1/\tau$. Because T_7 is highly symmetric, the tetrahedra T_8, T_9, T_{10} and T_{11} are obtained by appropriate subdivisions of T_7.

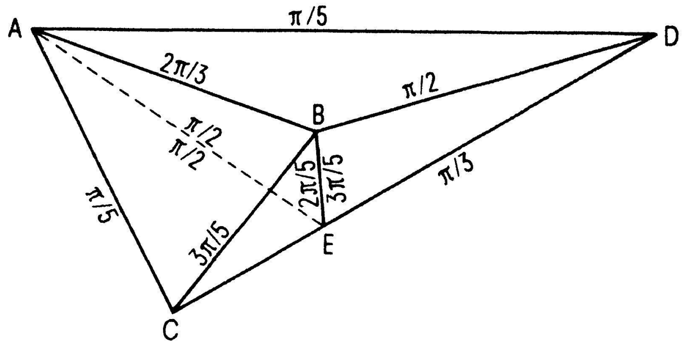

Fig. 3. $T_4(ABCD) = T_3(ABCE) + T_{13}(ABED)$

Similarly, T_{18} has an axis of symmetry. Hence, plane cuts through this axis produces two of T_{19} or two of T_{20}. Again, T_{23} has an axis of symmetry and it may be divided into two of T_{24} or two of T_{22}.

Note that in all the special tetrahedra, T_1 to T_{26}, most of the dihedral angles are rational fractions of π. When an angle is not rational, then there is always another angle of that tetrahedron, or two other angles to make their sum equal to π. Hence, the sum of the dihedral angles of each of these tetrahedra is a rational fraction of π. This is also true of Hill's tetrahedra of the first or second type.

However, the dihedral angles of Hill's third type do not conform to this condition. Their dihedral angles do not add to a rational fraction except when α is also a rational fraction of π.

In Table 1, α is a free variable. In Table II, α_2 and α_3 are the same as in [4]. The special values are given as follows.

$$\alpha_2 \approx 65^\circ\,, \qquad \tan\alpha_2 = \sqrt{9 - 2\sqrt{5}}\,,$$
$$\alpha_3 \approx 75^\circ\,, \qquad \tan\alpha_3 = \sqrt{9 + 2\sqrt{5}}\,,$$
$$\alpha_8 \approx 49^\circ\,,$$
$$\alpha_9 \approx 46^\circ 30'\,,$$
$$\alpha_{10} \approx 80^\circ 39'\,, \quad \cos\alpha_{10} = \sqrt{25 - 10\sqrt{5}}/10\,,$$
$$\alpha_{11} \approx 85^\circ\,,$$
$$\alpha_{12} \approx 37^\circ 46'\,, \quad \cos\alpha_{12} = \sqrt{10}/4\,.$$

M. Goldberg, Washington, D. C.

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