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A new proof of the reciprocity theorem for Dedekind sums

If

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{otherwise,} \end{cases}$$

the classical Dedekind sum $s(h, k)$ is defined by

$$s(h, k) = \sum_{n=1}^{k-1}{}' ((hn/k))((n/k)),$$

where k is a positive integer, and h is any integer. The most famous and useful property of these sums is the following result.

Reciprocity Theorem. If h and k are positive integers, and $(h, k) = 1$, then

$$s(h, k) + s(k, h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} + \frac{1}{hk}\right). \tag{1}$$

The beautiful, new monograph [1] by Rademacher and Grosswald presents four distinct proofs of (1). In this note, we give a completely new, short proof of (1).

Our proof is based on the classical Poisson summation formula. If f is of bounded variation on [a, b],

$$\frac{1}{2}\sum_{n=a}^{b}{}' \{f(n+0) + f(n-0)\} = \int_a^b f(x)\,dx + 2\sum_{n=1}^{\infty}\int_a^b f(x)\cos(2\pi n x)dx, \tag{2}$$

where the dash $'$ on the summation sign indicates that for the terms corresponding to $n = a$ and $n = b$, only $f(a+0)$ and $f(b-0)$, respectively, are counted. Let $a = 0$, $b = k$, and $f(x) = ((hx/k))((x/k))$ in (2). Observe that $f(0+0) = f(k-0) = 1/4$. On the right side of (2), let $x = ky$ in each of the integrals. An easy calculation [1, p. 23], yields

$$\int_0^k ((hx/k))((x/k))dx = k/12h.$$

Hence, (2) gives

$$\frac{1}{4} + s(h, k) = \frac{k}{12h} + 2k\sum_{n=1}^{\infty} I(k, h, n), \tag{3}$$

where

$$I(k, h, n) = \int_0^1 ((hy))((y))\cos(2\pi n k y)\,dy.$$

We now evaluate $I(k, h, n,)$. Write

$$I(k, h, n) = \sum_{r=o}^{h-1} \int_{r/h}^{(r+1)/h} \left(h y - r - \frac{1}{2}\right) \left(y - \frac{1}{2}\right) \cos (2 \pi n k y)\, dy$$

$$= \int_0^1 \left(h y - \frac{1}{2}\right) \left(y - \frac{1}{2}\right) \cos (2 \pi n k y)\, dy - \sum_{r=o}^{h-1} r \int_{r/h}^{(r+1)/h} \left(y - \frac{1}{2}\right) \cos (2 \pi n k y)\, dy$$

$$= \int_0^1 \left(h y - \frac{1}{2}\right) \left(y - \frac{1}{2}\right) \cos (2 \pi n k y)\, dy - \sum_{j=1}^{h-1} \int_{j/h}^{1} \left(y - \frac{1}{2}\right) \cos (2 \pi n k y)\, dy\,.$$

Integrating by parts, we obtain

$$I(k, h, n) = -\frac{1}{2 \pi n k} \int_0^1 \left\{2 h y - \frac{1}{2} (h+1)\right\} \sin (2 \pi n k y)\, dy$$

$$+ \frac{1}{2 \pi n k} \sum_{j=1}^{h-1} \left\{\left(\frac{j}{h} - \frac{1}{2}\right) \sin (2 \pi n k j/h) - \frac{1}{2 \pi n k} \left(1 - \cos (2 \pi n k j/h)\right)\right\}$$

$$= \frac{h}{(2 \pi n k)^2} + \frac{1}{2 \pi n k} \sum_{j=1}^{h-1} ((j/h)) \sin (2 \pi n k j/h) + \frac{1}{(2 \pi n k)^2} \sum_{j=0}^{h-1} \cos (2 \pi n k j/h).$$

The last sum on the right side above is 0 except when $n/h = \mu$, say, is a positive integer, in which case the sum is equal to h.

Using the above calculation, we find now that (3) yields

$$\frac{1}{4} + s(h, k) = \frac{k}{12 h} + \frac{h}{2 \pi^2 k} \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{4}$$

$$+ \sum_{j=1}^{h-1} ((j/h)) \sum_{n=1}^{\infty} \frac{\sin (2 \pi n k j/h)}{\pi n} + \frac{1}{2 \pi^2 h k} \sum_{\mu=1}^{\infty} \frac{1}{\mu^2}$$

$$= k/12 h + h/12 k - \sum_{j=1}^{h-1} ((j/h)) ((k j/h)) + 1/12 h k\,,$$

where we have used the well known Fourier series for $((x))$,

$$((x)) = -\sum_{n=1}^{\infty} \frac{\sin (2 \pi n x)}{\pi n}.$$

Equation (4) is clearly equivalent to (1), and so the proof of the reciprocity theorem is complete.

The ideas employed above can be applied to many generalizations of the Dedekind sums. An account of this work will be given elsewhere.

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REFERENCE

[1] H. RADEMACHER and E. GROSSWALD, *Dedekind Sums*, Carus Mathematical Monograph No. 16, Mathematical Association of America, Washington D.C., 1972.