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Setzen wir nun

$$z = \eta - \eta_0 + i\Theta, \quad z^* = \eta - \eta_0 - i\Theta,$$

so gewinnt die Gleichung (7.4) mit σ gemäss (7.5) die Gestalt (6.1), und zwar erhält man gerade den linearen Fall $\alpha_1(z) = iz$, $\alpha_2(z^*) = iz^*$. Für ν ergibt sich $\nu(\nu + 1) = -k_0/4$; man kann also ν derart wählen, dass man eine möglichst gute Näherung für die Poisson-Adiabate erhält.

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LITERATURVERZEICHNIS

[1] K. W. Bauer und E. Peschl, *Ein allgemeiner Entwicklungssatz für die Lösungen der Differentialgleichung* $(1 + \varepsilon z\bar{z})^2 w_{z\bar{z}} + \varepsilon n(n+1) w = 0$ *in der Nähe isolierter Singularitäten.* Sitz.-Ber. math.-naturw. Kl. Bayer. Akad. Wiss. München (1965).

[2] S. Bergman, *Integral Operators in the Theory of Linear Partial Differential Equations*, 2nd rev. print., (Berlin 1969).

[3] L. Bers, *An Outline of the Theory of Pseudoanalytic Functions*, Bull. Amer. Math. Soc. *62*, 291–331 (1956).

[4] S. V. Falkowitsch, *On the Theory of the Laval Nozzle*, Inst. Mech. Acad. Sci. USSR. Appl. Math. Mech. *10*, 503–512.

[5] G. Kneis, *Eine kanonische Gestalt für indefinite quadratische Differentialformen und globale Darstellungen für negativ gekrümmte Flächen im* R^3, Diss. (Halle 1971).

[6] M. Kracht und E. Kreyszig, *Bergman-Operatoren mit Polynomen als Erzeugenden*, Manuscripta math. *1*, 369–376 (1969).

[7] E. Kreyszig, *Die Realteil- und Imaginärteilflächen analytischer Funktionen*, El. Math. *24*, 25–31 (1968).

[8] E. Kreyszig und A. Pendl, *Über die Gauss-Krümmung der Real- und Imaginärteilflächen analytischer Funktionen*, El. Math. *28*, 10–13 (1973).

[9] H. P. Künzi, *Quasikonforme Abbildungen* (Berlin 1960).

[10] E. Lanckau, *Eine Anwendung der Bergmanschen Operatorenmethode auf Profilströmungen im Unterschall*, Wiss. Z. Techn. Univ. Dresden *8*, 200–207 (1958/59).

[11] R. v. Mises, *Mathematical Theory of Compressible Fluid Flow* (New York 1958).

[12] C. B. Morrey, *On the Solutions of Quasi-linear Elliptic Partial Differential Equations*, Trans. Amer. Math. Soc. *43*, 126–166 (1938).

[13] I. N. Vekua, *Verallgemeinerte analytische Funktionen* (Berlin 1963).

[14] I. N. Vekua, *New Methods for Solving Elliptic Equations* (New York 1967).

On a Cubic Functional Equation defined on Groups

I. Introduction

The following factorization yields the conditional identity $a^3 + b^3 + c^3 = 3abc$ for real numbers a, b and c with $a + b + c = 0$.

$$\begin{aligned} & a^3 + b^3 + c^3 - 3abc \\ &= (a+b+c)\,(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= \frac{1}{2}(a+b+c)\,[(a-b)^2 + (b-c)^2 + (c-a)^2] \\ &= (a+b+c)\,(a + b\omega + c\omega^2)\,(a + b\omega^2 + c\omega) \qquad (1) \end{aligned}$$

where $\omega = -(1/2)(1 - \sqrt{3}i)$ is a complex root of unity.

As an analogy, the following functional equation is proposed and studied

$$f^3(xy^{-1}) + f^3(yz^{-1}) + f^3(zx^{-1}) = 3f(xy^{-1})\,f(yz^{-1})\,f(zx^{-1}) \qquad (2)$$

where f is defined on an abelian or non-abelian group G and assumes its image in a field F. For simplicity, the notation $f^3(t)$ is used to denote $[f(t)]^3 = f(t) \cdot f(t) \cdot f(t)$.

Let Φ be a homomorphism from group G to F, with F viewed as an additive group. Then

$$\Phi(x y^{-1}) + \Phi(y z^{-1}) + \Phi(z x^{-1})$$
$$= [\Phi(x) - \Phi(y)] + [\Phi(y) - \Phi(z)] + [\Phi(z) - \Phi(x)] = 0\,,$$

and by (1), Φ satisfies (2). Not all solutions to (2) are group homomorphisms, since any constant function from G to F clearly satisfies (2). Moreover, constant functions are always continuous no matter what topologies are put on G and F. Continuity alone, therefore, may not be sufficient for a solution to be a homomorphism.

The question whether constant functions and group homomorphisms are the only solutions to (2) is answered in the negative by the first of the following two theorems which illustrate that the solution set of (2) depends on F as well as on G.

Before any theorem is proved, it is worthwhile to note that equation (2) on G is equivalent to the following functional equation on G:

$$f^3(x y^{-1}) + f^3(y) + f^3(x^{-1}) = 3 f(x y^{-1})\, f(y)\, f(x^{-1})\,. \tag{3}$$

Clearly (2) reduces to (3) after substituting the group identity e for z in (2). On the other hand, in (3) let $y = YZ^{-1}$ and $x^{-1} = ZX^{-1}$, then $xy^{-1} = XZ^{-1}ZY^{-1} = XY^{-1}$ and (2) is recaptured from (3).

II. Main Results

Theorem 1. Let $G = \{e, a, b, c\}$ be the Klein four-group with the identity e and $a^2 = b^2 = c^2 = e$, $ab = ba = c$, $bc = cb = a$, $ac = ca = b$. Let F be a field not of characteristic 2 or 3. Then the solution set of (2) consists of the following functions:

(i) constant functions.

(ii) functions f such that $f(e) = 2k$, $f(\pi a) = -k$, $f(\pi b) = -k$ and $f(\pi c) = 2k$, where k is an arbitrary element in F and π is a permutation on $\{a, b, c\}$.

(iii) functions f such that $f(e) = 2l$, $f(\pi a) = f(\pi b) = f(\pi c) = -l$ where l is an arbitrary element in F.

Proof. In (3) let $x = y \neq e$ then $xy = e$ and (3) reduces to

$$f^3(e) + 2 f^3(x) = 3 f(e)\, f^2(x),$$

from which it follows that for each $x \neq e$ in G

$$0 = 2 f^3(x) - 3 f(e)\, f^2(x) + f^3(e) = [2 f(x) + f(e)]\, [f(x) - f(e)]^2\,. \tag{4}$$

Thus, $f(a) = (-1/2)\, f(e)$ or $f(a) = f(e)$; $f(b) = (-1/2)\, f(e)$ or $f(b) = f(e)$; $f(c) = (-1/2)\, f(e)$ or $f(c) = f(e)$.

In each of the cases below, functions from G to F are defined to satisfy (4).

Case 1 $\quad f(\pi a) = -\dfrac{1}{2} f(e)\,, \quad f(\pi b) = f(\pi c) = f(e)$

Case 2 $\quad f(\pi c) = f(e) = 2k, \quad k \in F, \quad f(\pi a) = f(\pi b) = -\dfrac{1}{2} f(e) = -k$

Case 3 $\quad f(e) = 2l, \quad l \in F, \quad f(a) = f(b) = f(c) = -\dfrac{1}{2} f(e) = -l$

Case 4 $\quad f(a) = f(b) = f(c) = f(e)\,.$

If the functions shown in Case 1 are solutions to (2), then $f(e) = 0$ and $f(x) \equiv 0$ for each x in G. This can be seen by setting $x = a$ and $y = b$ in (3) and obtaining $0 = f^3(a) + f^3(b) + f^3(c) - 3\, f(a)\, f(b)\, f(c) = (-1/8 + 1 + 1 + 3/2)\, f^3(e)$. Notice that $f(x) \equiv 0$ is the only group homomorphism from G to F.

While constant functions defined in Case 4 are clearly solutions to (2), some classification of the relation between x and y in (3) is needed in order to verify that functions defined in Cases 2 and 3 all satisfy (3).

Case I.	$x = y$	Case II.	$x \neq y$
Subcase I-1	$x = y = e$	Subcase II-1	one of x, y is e
Subcase I-2	$x = y \neq e$	Subcase II-2	none of x, y is e.

Under Subcase I-1 (3) reduces to $f^3(e) + f^3(e) + f^3(e) = 3\, f^3(e)$ which is clearly satisfied by any function f from G to F.

Under Subcase I-2, when $x = y = \pi(a) \neq c$, or under Subcase II-1 when $x = e$ and $y = \pi(a)$, or $y = e$ and $x = \pi(a)$, (3) reduces to

$$f^3(e) + f^3[\pi(a)] + f^3[\pi(a)] = 3\, f(e)\, f[\pi(a)]\, f[\pi(a)]\,. \tag{5}$$

In Subcase II-2, when $x = \pi(a)$ and $y = \pi(b)$, (3) reduces to

$$f^3(a) + f^3(b) + f^3(c) = 3\, f(a)\, f(b)\, f(c)\,. \tag{6}$$

All these functions presented under Cases 2 and 3 are defined to satisfy (4) in general, and (5) in particular. A simple computation also shows that they satisfy (6). The proof is completed.

Theorem 2. Let G be a multiplicative group, and let F be a formally real field. If $f: G \to F$ is a solution to (2) such that $f(e) = 0$, where e is the identity in G, then $Z[f] = \{x \mid f(x) = 0\}$ is a normal subgroup of G. Moreover, if in $G/Z[f]$ the identity is the only element of order 3, then f is a group homomorphism. In particular, if $Z[f] = \{e\}$, then f is injective; and also, if e is the only element of order 3 in G, then f is an isomorphism.

Proof. Since $f(e) = 0$, setting $y = e$ in (3) yields

$$\begin{aligned} 0 &= f^3(x) + f^3(x^{-1}) \\ &= [f(x) + f(x^{-1})]\, [f^2(x) - f(x)\, f(x^{-1}) + f^2(x^{-1})] \\ &= \frac{1}{2}\, [f(x) + f(x^{-1})]\, \{[\, f(x) - f(x^{-1})]^2 + f^2(x) + f^2(x^{-1})\}\,. \end{aligned}$$

If $[f(x) - f(x^{-1})]^2 + f^2(x) + f^2(x^{-1}) = 0$, then $f(x) = f(x^{-1}) = 0$, since F is formally real. If $f(x) \neq 0$ or $f(x^{-1}) \neq 0$, then by the preceding factorization, $f(x) + f(x^{-1}) = 0$, which also holds for $f(x) = f(x^{-1}) = 0$. Thus

$$f(x) + f(x^{-1}) = 0, \quad x \in G\,. \tag{7}$$

From (7) it follows that $x \in Z[f]$ if and only if $x^{-1} \in Z[f]$. Now suppose that $x \in Z[f]$ and $y \in Z[f]$. Then, by (3), $f^3(xy) + f^3(y^{-1}) + f^3(x^{-1}) = 3\, f(xy)\, f(y^{-1})\, f(x^{-1}) = 0$. This implies that $f^3(xy) = 0$, $f(xy) = 0$, and $xy \in Z[f]$. Thus $Z[f]$ is a subgroup of G.

For the normality of $Z[f]$, it suffices to prove that for each x in G and for each z in $Z[f]$, there exists an element t in $Z[f]$ such that $xz = tx$. Set $t = xzx^{-1}$, clearly, $tx = (xzx^{-1})x = xz$. To prove $t \in Z[f]$, observe that

$$f^3(xzx^{-1}) + f^3(x) + f^3(z^{-1}x^{-1}) = 3f(xzx^{-1})f(x)f(z^{-1}x^{-1}) \tag{8}$$

which is a direct consequence of (3). Moreover, by (3) again, $f^3(z^{-1}x^{-1}) + f^3(x) + f^3(z) = 3f(z^{-1}x^{-1})f(x)f(z)$, which reduces to $f^3(z^{-1}x^{-1}) + f^3(x) = 0$ and $-f^3(xz) + f^3(x) = 0$, since $f(z) = 0$ and $f(z^{-1}x^{-1}) = -f(xz)$.

Through factorization,

$$f^3(x) - f^3(xz) = [f(x) - f(xz)][f^2(x) + f(x)f(xz) + f^2(xz)]$$

$$= [f(x) - f(xz)][f^2(x) - f(x)f(z^{-1}x^{-1}) + f^2(z^{-1}x^{-1})]$$

$$= \frac{1}{2}[f(x) - f(xz)]\{f^2(x) + f^2(z^{-1}x^{-1}) + [f(x) - f(z^{-1}x^{-1})]^2\}.$$

If the second factor is zero, then $f(x) = f(z^{-1}x^{-1}) = f(x) - f(z^{-1}x^{-1}) = 0$, i.e., $f(x) = f(z^{-1}x^{-1}) = 0$. If $f(x) \neq 0$ or $f(z^{-1}x^{-1}) \neq 0$, then the first factor must be zero, and thus $f(x) = f(xz)$ which also holds for $f(x) = f(z^{-1}x^{-1}) = 0$. Now back to (8):

$$f^3(xzx^{-1}) = -3f(xzx^{-1})f(x)f(xz) = -3f(xzx^{-1})f^2(x), \qquad \text{i.e.,}$$

$$f(xzx^{-1})[f^2(xzx^{-1}) + 3f^2(x)] = 0.$$

This implies that either $f(xzx^{-1}) = 0$ or $f^2(xzx^{-1}) + 3f^2(x) = 0$. In both cases, $f(xzx^{-1}) = 0$. Thus $xzx^{-1} \in Z[f]$ and $Z[f]$ is a normal subgroup.

Next, it will be shown that the cosets of G modulo $Z[f]$ coincide with the equivalence classes of the relation defined by $x \sim y$ if and only if $f(x) = f(y)$. Now assume that $f(x) = f(y)$. By (3) and (7), $f^3(x^{-1}y) + f^3(y^{-1}) + f^3(x) = 3f(x^{-1}y)f(y^{-1})f(x) = f^3(x^{-1}y) - f^3(y) + f^3(x) = -3f(x^{-1}y)f(y)f(x)$. Now that $f(x) = f(y)$, therefore, $f^3(x^{-1}y) + 3f(x^{-1}y)f^2(x) = 0$ and $f(x^{-1}y)[f^2(x^{-1}y) + 3f^2(x)] = 0$. This implies that either $f(x^{-1}y) = 0$ or $f(x^{-1}y) = f(x) = 0$. In both cases $x^{-1}y \in Z[f]$. On the other hand, $y = x(x^{-1}y)$, hence $y \in \hat{x}$ and $\hat{x} = \hat{y}$, where $\hat{x}$ denotes the coset of x modulo $Z[f]$. In particular, if $f(x) = f(y)$ and if $Z[f] = \{e\}$, then $f(x^{-1}y) = 0$, $x^{-1}y = e$ and $x = y$. In this case, f is a one-to-one function.

Conversely, suppose $\hat{x} = \hat{y}$ i.e., $y = xu$ for some $u \in Z[f]$. Then, by (3), $f^3(xu) + f^3(u^{-1}) + f^3(x^{-1}) = 3f(xu)f(u^{-1})f(x^{-1}) = 0$. This yields $0 = f^3(y) - f^3(x) = (1/2)[f(y) - f(x)]\{f^2(x) + f^2(y) + [f(x) + f(y)]^2\}$. Therefore, either $f(x) = f(y)$ or $f(x) = f(y) = 0$. Thus $\hat{x} = \hat{y}$ implies that $f(x) = f(y)$.

In proving that f is an isomorphism under the conditions that $Z[f] = \{e\}$ and e is the only element of order 3 in G, rewrite (3) to obtain

$$0 = f^3(xy) + f^3(y^{-1}) + f^3(x^{-1}) - 3f(xy)f(y^{-1})f(x^{-1})$$

$$= \frac{1}{2}[f(xy) + f(y^{-1}) + f(x^{-1})] \cdot$$

$$([f(xy) - f(y^{-1})]^2 + [f(y^{-1}) - f(x^{-1})]^2 + [f(x^{-1}) - f(xy)]^2).$$

If the second factor is 0, then $f(xy) = f(y^{-1}) = f(x^{-1})$. Here f is proved to be injective, therefore $xy = y^{-1} = x^{-1}$. Hence $x^2y = xx^{-1} = e$. This implies $x = e = y$, since e is the only element of order 3 in G. If $x \neq e$ or $y \neq e$, then the first factor in the preceding factorization must be 0, which also holds when $x = e$ and $y = e$. Thus, $f(xy) = -f(x^{-1}) - f(y^{-1}) = f(x) + f(y)$ by (7), and f is an isomorphism from G to F.

To prove that f is a homomorphism under the assumption that $\hat{e}$ is the only element of order 3 in $G/Z[f]$, define a function $\hat{f} : G/Z[f] \to F$ by $\hat{f}(\hat{x}) = f(x)$. Since $x^{-1}y \in Z[f]$ if and only if $f(x) = f(y)$, $\hat{f}$ is well-defined and injective. Clearly, f factors through $G/Z[f]$ in the sense that $f = \hat{f} \circ q$ where q is the quotient map. When f satisfies (3) with $f(e) = 0$, $\hat{f}$ with $\hat{f}(\hat{e}) = 0$ satisfies (3) defined on $G/Z[f]$ i.e.,

$$\hat{f}^3(\hat{x}\hat{y}^{-1}) + \hat{f}^3(\hat{y}) + \hat{f}^3(\hat{x}^{-1}) = 3\,\hat{f}(\hat{x}\hat{y}^{-1})\,\hat{f}(\hat{y})\,\hat{f}(\hat{x}^{-1}) .$$

Since it is assumed that $\hat{e}$ is the only element of order 3 in $G/Z[f]$, hence, by the preceding proof, $\hat{f}$ is an isomorphism from $G/Z[f]$ to F. This in turn implies that $f = \hat{f} \circ q$ is a homomorphism from G to F. The proof is completed.

III. Remarks

Theorem 2 enables us to determine solutions for (2) on G if f has a preassigned set of zeros containing e. Some applications are in order. Let R^+ be the multiplicative group of positive reals and R be the field of real numbers. Functions $f: R^+ \to R$ satisfying (2) with $Z[f] = \{1\}$ abound. Clearly, for each non-zero k, $f(x) = k \log x$ is such a one. On the other hand, let R_0 denote the multiplicative group of non-zero reals. Consider functions $f: R_0 \to R$ satisfying (2) with preassigned $Z[f] = \{-1,1\}$. Then obviously $R_0/Z[f]$ is isomorphic to R^+ and each function $\hat{f}: R_0/Z[f] \to R$ satisfying (3) is an isomorphism. If $\hat{f}$ is taken to be $\hat{f}(x) = k \log x$ with $k \neq 0$ and $x > 0$, then $f = (\hat{f} \circ q) : R_0 \to R$ can simply be $f(x) = k \log |x|$, which clearly satisfies (2) with $Z[f] = \{-1,1\}$. Theorem 2 can be useful in other directions. Suppose that G is an abelian group such that each element except the identity e has infinite order. Let p be a prime number different from 3. Then $H = \{x^p \mid x \in G\}$ is a subgroup of G and each element in G/H is of order p. There is no isomorphism from G/H to a formally real field F viewed as an additive group. Hence there does not exist any function $f: G \to F$ satisfying (2) with $Z[f] = H$, otherwise, such an f induces an isomorphism $\hat{f}$ by $f = \hat{f} \circ q$ which yields a contradiction.

The study of (2) on groups is far from complete. Some other observation is noted in the following. With each element a in G, an inner automorphism is associated such that x in G is mapped to axa^{-1}. If f is a solution to (2), then f_a is also a solution, where f_a is defined by $f_a(x) = f(axa^{-1})$, f_a may not be equal to f if G is non-commutative. The mapping $(a, f) \to f_a$ defines an operation of G on the solution set S of (2). For each f in S, the set $\{a \mid a \in G, f_a = f\}$ is a subgroup of G and is called the isotropy group of f in G. The set $\{f_a \mid a \in G\}$ is called the orbit of f under G. Orbits and isotropy groups may be useful in the study of (2).

In Theorem 2, if the set of complex numbers is taken as the field F, then the following factorization holds when $f(e) = 0$.

$$\begin{aligned} 0 &= f^3(x) + f^3(x^{-1}) = [f(x) + f(x^{-1})]\,[f^2(x) - f(x)\,f(x^{-1}) + f^2(x^{-1})] \\ &= [f(x) + f(x^{-1})]\,[f(x) + \omega\, f(x^{-1})]\,[f(x) + \omega^2\, f(x^{-1})] . \end{aligned}$$

From this it follows that $f(x) = -f(x^{-1})$ or $f(x) = -\omega f(x^{-1})$ or $f(x) = -\omega^2 f(x^{-1})$.

It is not necessary that $f(x) = -f(x^{-1})$ for each x in G. This shows some of the difficulties in the problem of determining the solution of (2) when f is complex-valued.

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REFERENCES

[1] J. Aczél, *Lectures on Functional Equations and their Applications* (Academic Press, New York-London 1966).

[2] N. Jacobson, *Lectures in Abstract Algebra*, Vol. 3 (Van Nostrand, Princeton, N. J. 1964).

[3] O. Taussky, *Sums of Squares*, Amer. Math. Monthly 77 (1970), 805–830.

Kleine Mitteilungen

Über die Chordalkurve zweier Kegelschnitte

Die Hüllkurve der Geraden, die aus zwei Kegelschnitten k_1, k_2 Sehnen gleicher Länge ausschneiden, heisse die Chordalkurve von k_1 und k_2.

1. **Die Chordalparabel zweier Kreise** k_1, k_2 (Mittelpunkte K_1, K_2, Radien r_1, r_2). Es sei $K_1 \neq K_2$. Mittelpunkt von $K_1 K_2$ sei F. Die Chordale von k_1 und k_2 sei s. Eine Gerade g habe von K_1 bzw. K_2 die Abstände p_1 bzw. p_2. Die Normale aus F auf g schliesse mit $K_1 K_2$ den Winkel α ein und schneide s im Punkt G. Es ist $\overline{FG} = (p_1 + p_2)/2$. Die von g aus k_1 und k_2 geschnittenen Sehnen sind gleich lang, wenn $r_1^2 - p_1^2 = r_2^2 - p_2^2$ ist. Daraus folgt $(p_2 + p_1)(p_2 - p_1) = r_2^2 - r_1^2$ oder $2 \cdot \overline{FG} \cdot \overline{K_1 K_2} \cdot \cos\alpha = r_2^2 - r_1^2$; daher ist $\overline{FG} \cdot \cos\alpha$ konstant. Alle G liegen auf s, denn für $\alpha = 0$ ist $g \equiv s$. Es folgt:

Satz 1. *Alle Geraden, die die (nichtkonzentrischen) Kreise k_1, k_2 nach längengleichen Sehnen schneiden, umhüllen für $r_1 \neq r_2$ die «Chordalparabel» p von k_1 und k_2 (Brennpunkt F, Scheiteltangente s). p berührt auch die gemeinsamen Tangenten von k_1 und k_2.* Für $r_1 = r_2$ zerfällt p (als Klassenkurve) in das Strahlbüschel F und in das Büschel der zu $K_1 K_2$ parallelen Geraden.

g schneidet nur dann reelle Sehnen aus k_1 und k_2, wenn $|r_1 - r_2| \leqq 2 \cdot \overline{FG} \leqq r_1 + r_2$ ist; die Intervallgrenzen gehören zu den gemeinsamen Tangenten von k_1 und k_2; existieren 4 bzw. 2 bzw. 0 reelle gemeinsame Tangenten von k_1 und k_2, so gibt es 2 bzw. 1 bzw. 0 Bögen auf p, deren Punkte Tangenten g mit reellen Sehnen von k_1 und k_2 besitzen.

Da p durch F und s bestimmt ist, gilt die Umkehrung von Satz 1:

Satz 2. *Eine Parabel p ist Chordalparabel je zweier Kreise, die sich auf der Scheiteltangente von p schneiden und deren Mitten auf der Parabelachse symmetrisch zum Brennpunkt von p liegen.*

2. **Die Chordalgeraden dreier Kreise** k_i (Mitten K_i, Radien r_i). Die Chordalparabel von k_i und k_j sei p_{ij} (Brennpunkt F_{ij} = Mitte von $K_i K_j$, Scheiteltangente s_{ij} = Chordale von k_i und k_j). Eine eigentliche gemeinsame Tangente von p_{12} und p_{13} schneidet k_1 und k_2, ebenso k_1 und k_3, daher auch k_2 und k_3 nach längengleichen Sehnen. Sie berührt also auch p_{23}. Eine solche Gerade, die die k_i nach drei längengleichen Sehnen schneidet, heisse eine Chordalgerade der k_i.