A triangle transformation

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stehende Polytop ist wiederum ein reguläres 16-Zell mit der Kantenlänge $\sqrt{2}$, also den beiden anderen kongruent.

Beide Beispiele stehen nach H. Groemer [3] in engem Zusammenhang damit, dass es Parkettierungen des E_4 mit lauter regulären 24-Zellen (alle in gleicher Drehlage) bzw. mit lauter regulären 16-Zellen (in drei verschiedenen Drehlagen) gibt (vgl. H. S. M. Coxeter [1], S. 296). Da solche Parkettierungen mit regulären 120-Zellen nicht existieren, ist es vermutlich schwierig, die Zerlegungsgleichheit des 120-Zells mit einem Hyperwürfel explizit zu realisieren.

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A Triangle Transformation

1. The configuration of a triangle on the sides of which polygons of a certain kind are described is a much studied theme in elementary geometry. The following variant does not seem to be well-known.

On the sides of a given triangle ABC similar isosceles triangles BCA, CAB, ABC, are constructed (Fig. 1), with bases BC, CA, AB, all outward or all inward, the base angle φ being taken positive or negative respectively ($-\pi/2 < \varphi < \pi/2$). The operation thus defined which transforms the triangle $\Delta = ABC$ into $\Delta_1 = A_1B_1C_1$ will be denoted by $T(\varphi)$.

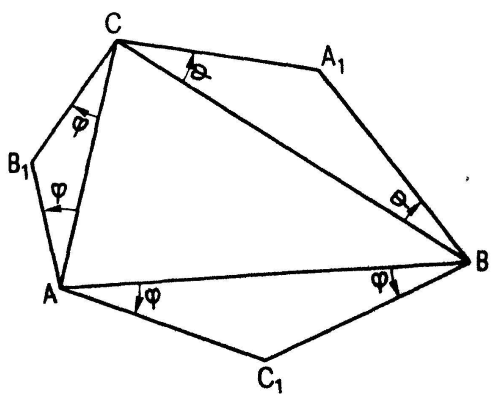

Fig. 1

The sides of Δ, Δ_1 are a, b, c and a_1, b_1, c_1, the oriented areas F and F_1, $S = a^2 + b^2 + c^2$, $S_1 = a_1^2 + b_1^2 + c_1^2$. As $A_1 B = A_1 C = (1/2)\, a \cos^{-1} \varphi$ etc., the cosine rule gives

$$\begin{aligned} 4 a_1^2 &= \cos^{-2} \varphi \{b^2 + c^2 - 2 b c \cos (\alpha + 2 \varphi)\} \\ &= \cos^{-2} \varphi \{b^2 + c^2 - (b^2 + c^2 - a^2)\ (1 - 2\sin^2\varphi) + 2 b c \sin\alpha \sin 2\varphi\} \\ &= (2 b^2 + 2 c^2 - a^2)\, t^2 + a^2 + 8 F t\,, \end{aligned} \tag{1.1}$$

with $t = \tan\varphi$. For $4 b_1^2$ and $4 c_1^2$ we obtain analogous formulas. Hence

$$S_1 = \frac{1}{4}\,(3 t^2 + 1)\ S + 6\, t\, F\,. \tag{1.2}$$

For the area of Δ_1 we have

$$F_1 = F + \sum (\text{area } B A_1 C) - \sum (\text{area } B_1 A C_1)\,. \tag{1.3}$$

Obviously

$$\text{area } B A_1 C = \frac{1}{4}\, t\, a^2\,, \tag{1.4}$$

and

$$\begin{aligned} \text{area } B_1 A C_1 &= \frac{1}{8} \cos^{-2}\varphi \cdot bc\ \sin\,(\alpha + 2\varphi) \\ &= \frac{1}{8} \cos^{-2}\varphi \{2 F\,(\cos^2\varphi - \sin^2\varphi) + (b^2 + c^2 - a^2)\ \sin\varphi \cos\varphi\} \\ &= \frac{1}{4}\ (1 - t^2)\ F + \frac{1}{8}\ t\,(b^2 + c^2 - a^2)\,. \end{aligned} \tag{1.5}$$

Hence from 1.4 and 1.5

$$\text{area } B A_1 C - \text{area } B_1 A C_1 = \frac{1}{4}\ (t^2 - 1)\ F + \frac{1}{8}\, t\,(3 a^2 - b^2 - c^2)\,,$$

and 1.3 gives us

$$F_1 = \frac{1}{4}\,(3\, t^2 + 1)\ F + \frac{1}{8}\, t\, S \tag{1.6}$$

The formulas (1.2) and (1.6) express F_1 and S_1 as linear functions of F and S. They have been derived recently in a different context, for $\varphi > 0$ only and with a less simple proof of (1.6) by Toscano [1]. They are valid for all values of φ; if F_1 and F have different signs the orientation of Δ_1 is the opposite of that of Δ.

2. F and S are related to the well-known Brocard's angle ω of the triangle [2]. We have, accepting a negative value of ω for a triangle with negative orientation

$$\cot \omega = S/4 F \tag{2.1}$$

One always has $\cot^2 \omega \geqq 3$, or $-\pi/6 \leqq \omega \leqq \pi/6$, with equality only for equilateral triangles.

From (1.2) and (1.6) it follows

$$\cot\, \omega_1 = \frac{(3\, t^2 + 1) \cot\, \omega + 6\, t}{2\, t \cot \omega + (3\, t^2 + 1)}\,, \tag{2.2}$$

which means that ω_1 depends only on t and ω. *If two triangles are equibrocardian their transforms by $T(\varphi)$ are equibrocardian as well.*

The right-hand side of (2.2) is a linear function of cot ω the determinant of which is $(3t^2-1)^2$. Hence it is singular only if $t = \pm\sqrt{1/3}$; then for any ω we have cot $\omega_1 = \pm\sqrt{3}$. Therefore *by $T(\pm\pi/6)$ any triangle is tranformed into an equilateral triangle,* a well-known theorem.

We divide the set of all triangles into classes $K(\omega)$, the elements of a class being the triangles with a given ω. The classes $K(\pm\pi/6)$ contain the equilateral triangles, $K(0)$ is the class of degenerated triangles. The conclusion is: any $T(\varphi)$ permutes the classes $K(\omega)$.

(2.2) may be written as follows

$$(3t^2+1)\ (\cot\omega_1 - \cot\omega) + 2t\ \cot\omega_1\ \cot\omega - 6t = 0 \tag{2.3}$$

$\omega_1 = \omega$ implies

$$t\,(\cot^2\omega - 3) = 0 \tag{2.4}$$

Hence the two classes of equilateral triangles are invariant for any $T(\varphi)$ and any class $K(\omega)$ is invariant for $T(0)$. Both properties are obvious: $T(0)$ transforms a triangle into that of the midpoints of its sides.

If ω and ω_1 are given, (2.3) is a quadratic equation for t:

$$3t^2(\cot\omega_1 - \cot\omega) + 2t\,(\cot\omega_1\cot\omega - 3) + (\cot\omega_1 - \cot\omega) = 0 \tag{2.5}$$

Its discriminant d satisfies

$$d = (\cot^2\omega_1 - 3)\ (\cot^2\omega - 3) \gtreqless 0 \tag{2.6}$$

Hence 2.5 has real roots t_1, t_2; *in general, two transformations $T(\varphi_1)$, $T(\varphi_2)$ exist which transform a given class $K(\omega)$ into a given class $K(\omega_1)$.* One always has $t_1 t_2 = 1/3$.

If t_0 is a root of (2.5) then $-t_0$ is one of the equation with ω and ω_1 interchanged. *If $T(t)$ transforms $K(\omega)$ into $K(\omega_1)$ the two transforming $K(\omega_1)$ into $K(\omega)$ are $T(-t)$ and $T(-1/3t)$.* In particular: Δ and $T(-t)\,\{T(t)\cdot\Delta\}$ are equibrocardian.

If $T(t_1)\cdot\Delta = \Delta_1$ and $T(t_2)\cdot\Delta_1 = \Delta_2$, we have

$$\cot\omega_2 = \frac{\{3(t_1+t_2)^2 + (3t_1t_2+1)^2\}\cos\omega + 6(t_1+t_2)(3t_1t_2+1)}{2(t_1+t_2)(3t_1t_2+1)\cos\omega + \{3(t_1+t_2)^2 + (3t_1t_2+1)^2\}}$$

or, if $3t_1t_2 + 1 \neq 0$,

$$\cos\omega_2 = \frac{(3t_{12}^2+1)\cos\omega + 6t_{12}}{2t_{12}\cos\omega + (3t_{12}^2+1)}, \tag{2.7}$$

with

$$t_{12} = \frac{t_1+t_2}{3t_1t_2+1}, \tag{2.8}$$

giving *the multiplication rule* for two transformations of the set: $T(t_2)\cdot\{T(t_1)\cdot\Delta\}$ and $T(t_{12})\cdot\Delta$ are equibrocardian triangles. The multiplication is *commutative.* If Δ is a given triangle with Brocard angle ω there are two transformations T such that $\omega_1 = 0$ and hence cot $\omega_1 = \infty$. From (2.5) it follows that they are $T(t_i)$, $i = 1,2$, such that t_i are the roots of

$$3t^2 + 2t\cot\omega + 1 = 0, \tag{2.9}$$

that is

$$t_i = \frac{1}{3}\{-\cot\omega \pm (\cot^2\omega - 3)^{1/2}\}. \qquad (2.10)$$

If on the sides of a triangle we construct isosceles triangles with $\tan\varphi = t_1$ or t_2 the points A_1, B_1, C_1 are collinear. As $t_1 t_2 = 1/3$, $t_1 + t_2 = (-2/3)\cot\omega$ we have $\varphi_1 + \varphi_2 = \omega - \pi/2$.

3. Properties of the transformation $T(t)$ may also be found by vector algebra. We denote the point P of the plane by the vector $\overline{P}$ from the origin O to P; the unit vector perpendicular to the plane by $\bar{e}$.

We obtain

$$2\overline{A}_1 = (\overline{B} + \overline{C}) + t\bar{e} \times (\overline{B} - \overline{C}), \; 2\overline{B}_1 = (\overline{C} + \overline{A}) + t\,\bar{e} \times (\overline{C} - \overline{A}),$$
$$2\overline{C}_1 = (\overline{A} + \overline{B}) + t\bar{e} \times (\overline{A} - \overline{B}) \qquad (3.1)$$

As $\overline{A}_1 + \overline{B}_1 + \overline{C}_1 = \overline{A} + \overline{B} + \overline{C}$, we conclude: *the triangles Δ and $T\cdot\Delta$ have the same centroid.*

Furthermore if $\Delta_1 = T(t_1)\cdot\Delta$ amd $\Delta_2 = T(t_2)\cdot\Delta_1$ we have

$$2\overline{A}_2 = (\overline{B}_1 + \overline{C}_1) + t_2\bar{e} \times (\overline{B}_1 - \overline{C}_1)$$

or

$$4\overline{A}_2 = (2\overline{A} + \overline{B} + \overline{C}) + (t_1 + t_2)\,\bar{e} \times (\overline{C} - \overline{B}) + t_1 t_2 (2\overline{A} - \overline{B} - \overline{C}), \qquad (3.2)$$

and analogously for $4\overline{B}_2$ and $4\overline{C}_2$. If we take the origin O at the centroid G of ABC we obtain

$$4\overline{A}_2 = (1 + 3t_1t_2)\,\overline{A} + (t_1 + t_2)\,\bar{e} \times (\overline{C} - \overline{B}). \qquad (3.3)$$

Two special cases are of interest. If $t_1 + t_2 = 0$, $-t_2 = +t_1 = t$ one has

$$4\overline{A}_2 = (1 - 3t^2)\,\overline{A}, \; 4\overline{B}_2 = (1 - 3t^2)\,\overline{B}, \; 4\overline{C}_2 = (1 - 3t^2)\,\overline{C}. \qquad (3.4)$$

Therefore: *If on the sides of ABC we describe outward (inward) isosceles triangles with base angle φ and then on those of $A_1B_1C_1$ inward (outward) isosceles triangles with the same angle φ then $A_2B_2C_2$ and ABC are homothetic with respect to the centroid G, with the scale factor* $(1/4)\,(1 - 3\tan^2\varphi)$. The two triangles are congruent if $\tan\varphi = \pm\sqrt{5/3}$.

If $1 + 3t_1t_2 = 0$, $t_1 + t_2 = t$ we obtain

$$4\overline{A}_2 = t\bar{e} \times (\overline{C} - \overline{B}), \quad 4\overline{B}_2 = t\bar{e} \times (\overline{A} - \overline{C}), \quad 4\overline{C}_2 = t\bar{e} \times (\overline{B} - \overline{A}). \qquad (3.5)$$

Hence the medians of $A_2B_2C_2$ are perpendicular to the sides of ABC and equal to $(3/8)\,ta$, $(3/8)\,tb$, $(3/8)\,tc$; from this is follows that the sides of $A_2B_2C_2$ are proportional to the medians of ABC with ratio $t/2$. O. Bottema, Delft

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