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stehende Polytop ist wiederum ein reguläres 16-Zell mit der Kantenlänge  $\sqrt{2}$ , also den beiden anderen kongruent.

Beide Beispiele stehen nach H. Groemer [3] in engem Zusammenhang damit, dass es Parkettierungen des  $E_4$  mit lauter regulären 24-Zellen (alle in gleicher Drehlage) bzw. mit lauter regulären 16-Zellen (in drei verschiedenen Drehlagen) gibt (vgl. H. S. M. Coxeter [1], S. 296). Da solche Parkettierungen mit regulären 120-Zellen nicht existieren, ist es vermutlich schwierig, die Zerlegungsgleichheit des 120-Zells mit einem Hyperwürfel explizit zu realisieren.

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#### LITERATURVERZEICHNIS

- [1] H. S. M. COXETER, *Regular Polytopes*, (Methuen, London 1948).
- [2] M. DEHN, *Über den Rauminhalt*, Math. Ann. 55, 465–478 (1901).
- [3] H. GROEMER, *Über Würfel- und Raumzerlegungen*, El. Math. 19, 25–27 (1964).
- [4] H. HADWIGER, *Translationsinvariante, additive und schwachstetige Polyederfunktionale*, Arch. Math. 3, 387–394 (1952).
- [5] H. HADWIGER, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer-Verlag 1957.
- [6] B. JESSEN, *Zur Algebra der Polytope*, Nachr. Akad. Wiss. Göttingen II 1972, 47–53.
- [7] B. JESSEN, und A. THORUP, *The Algebra of Polytopes in Affine Spaces* (1973, noch unpubliziert).
- [8] H. LEBESGUE, *Sur les subdivisions des polyèdres réguliers en polyèdres réguliers*, Publ. math. Univ. Belgrade 6-7, 183–188 (1938).
- [9] H. LEBESGUE, *Sur l'équivalence des polyèdres, en particulier des polyèdres réguliers, et sur la dissection des polyèdres réguliers en polyèdres réguliers*, Ann. Soc. Polon. Math. 17, 193–226 (1939).

## A Triangle Transformation

1. The configuration of a triangle on the sides of which polygons of a certain kind are described is a much studied theme in elementary geometry. The following variant does not seem to be well-known.

On the sides of a given triangle  $ABC$  similar isosceles triangles  $BCA_1$ ,  $CAB_1$ ,  $ABC_1$  are constructed (Fig. 1), with bases  $BC$ ,  $CA$ ,  $AB$ , all outward or all inward, the base angle  $\varphi$  being taken positive or negative respectively ( $-\pi/2 < \varphi < \pi/2$ ). The operation thus defined which transforms the triangle  $\Delta = ABC$  into  $\Delta_1 = A_1B_1C_1$  will be denoted by  $T(\varphi)$ .

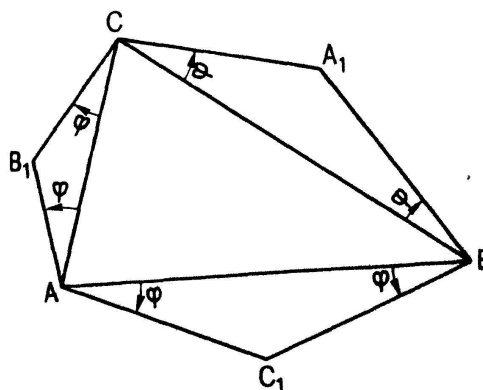


Fig. 1

The sides of  $\Delta$ ,  $\Delta_1$  are  $a, b, c$  and  $a_1, b_1, c_1$ , the oriented areas  $F$  and  $F_1$ ,  $S = a^2 + b^2 + c^2$ ,  $S_1 = a_1^2 + b_1^2 + c_1^2$ . As  $A_1B = A_1C = (1/2) a \cos^{-1} \varphi$  etc., the cosine rule gives

$$\begin{aligned} 4a_1^2 &= \cos^{-2} \varphi \{b^2 + c^2 - 2bc \cos(\alpha + 2\varphi)\} \\ &= \cos^{-2} \varphi \{b^2 + c^2 - (b^2 + c^2 - a^2)(1 - 2\sin^2 \varphi) + 2bc \sin \alpha \sin 2\varphi\} \\ &= (2b^2 + 2c^2 - a^2)t^2 + a^2 + 8Ft, \end{aligned} \quad (1.1)$$

with  $t = \tan \varphi$ . For  $4b_1^2$  and  $4c_1^2$  we obtain analogous formulas. Hence

$$S_1 = \frac{1}{4} (3t^2 + 1) S + 6tF. \quad (1.2)$$

For the area of  $\Delta_1$  we have

$$F_1 = F + \sum (\text{area } BA_1C) - \sum (\text{area } B_1AC_1). \quad (1.3)$$

Obviously

$$\text{area } BA_1C = \frac{1}{4} t a^2, \quad (1.4)$$

and

$$\begin{aligned} \text{area } B_1AC_1 &= \frac{1}{8} \cos^{-2} \varphi \cdot bc \sin(\alpha + 2\varphi) \\ &= \frac{1}{8} \cos^{-2} \varphi \{2F(\cos^2 \varphi - \sin^2 \varphi) + (b^2 + c^2 - a^2) \sin \varphi \cos \varphi\} \\ &= \frac{1}{4} (1 - t^2) F + \frac{1}{8} t (b^2 + c^2 - a^2). \end{aligned} \quad (1.5)$$

Hence from 1.4 and 1.5

$$\text{area } BA_1C - \text{area } B_1AC_1 = \frac{1}{4} (t^2 - 1) F + \frac{1}{8} t (3a^2 - b^2 - c^2),$$

and 1.3 gives us

$$F_1 = \frac{1}{4} (3t^2 + 1) F + \frac{1}{8} t S \quad (1.6)$$

The formulas (1.2) and (1.6) express  $F_1$  and  $S_1$  as linear functions of  $F$  and  $S$ . They have been derived recently in a different context, for  $\varphi > 0$  only and with a less simple proof of (1.6) by Toscano [1]. They are valid for all values of  $\varphi$ ; if  $F_1$  and  $F$  have different signs the orientation of  $\Delta_1$  is the opposite of that of  $\Delta$ .

2.  $F$  and  $S$  are related to the well-known Brocard's angle  $\omega$  of the triangle [2]. We have, accepting a negative value of  $\omega$  for a triangle with negative orientation

$$\cot \omega = S/4F \quad (2.1)$$

One always has  $\cot^2 \omega \geq 3$ , or  $-\pi/6 \leq \omega \leq \pi/6$ , with equality only for equilateral triangles.

From (1.2) and (1.6) it follows

$$\cot \omega_1 = \frac{(3t^2 + 1) \cot \omega + 6t}{2t \cot \omega + (3t^2 + 1)}, \quad (2.2)$$

which means that  $\omega_1$  depends only on  $t$  and  $\omega$ . *If two triangles are equibrocardian their transforms by  $T(\varphi)$  are equibrocardian as well.*

The right-hand side of (2.2) is a linear function of  $\cot \omega$  the determinant of which is  $(3t^2 - 1)^2$ . Hence it is singular only if  $t = \pm \sqrt{1/3}$ ; then for any  $\omega$  we have  $\cot \omega_1 = \pm \sqrt{3}$ . Therefore by  $T(\pm \pi/6)$  any triangle is transformed into an equilateral triangle, a well-known theorem.

We divide the set of all triangles into classes  $K(\omega)$ , the elements of a class being the triangles with a given  $\omega$ . The classes  $K(\pm \pi/6)$  contain the equilateral triangles,  $K(0)$  is the class of degenerated triangles. The conclusion is: any  $T(\varphi)$  permutes the classes  $K(\omega)$ .

(2.2) may be written as follows

$$(3t^2 + 1)(\cot \omega_1 - \cot \omega) + 2t \cot \omega_1 \cot \omega - 6t = 0 \quad (2.3)$$

$\omega_1 = \omega$  implies

$$t(\cot^2 \omega - 3) = 0 \quad (2.4)$$

Hence the two classes of equilateral triangles are invariant for any  $T(\varphi)$  and any class  $K(\omega)$  is invariant for  $T(0)$ . Both properties are obvious:  $T(0)$  transforms a triangle into that of the midpoints of its sides.

If  $\omega$  and  $\omega_1$  are given, (2.3) is a quadratic equation for  $t$ :

$$3t^2(\cot \omega_1 - \cot \omega) + 2t(\cot \omega_1 \cot \omega - 3) + (\cot \omega_1 - \cot \omega) = 0 \quad (2.5)$$

Its discriminant  $d$  satisfies

$$d = (\cot^2 \omega_1 - 3)(\cot^2 \omega - 3) \geq 0 \quad (2.6)$$

Hence 2.5 has real roots  $t_1, t_2$ ; in general, two transformations  $T(\varphi_1), T(\varphi_2)$  exist which transform a given class  $K(\omega)$  into a given class  $K(\omega_1)$ . One always has  $t_1 t_2 = 1/3$ .

If  $t_0$  is a root of (2.5) then  $-t_0$  is one of the equation with  $\omega$  and  $\omega_1$  interchanged. If  $T(t)$  transforms  $K(\omega)$  into  $K(\omega_1)$  the two transforming  $K(\omega_1)$  into  $K(\omega)$  are  $T(-t)$  and  $T(-1/3t)$ . In particular:  $\Delta$  and  $T(-t)\{\Delta\}$  are equibrocardian.

If  $T(t_1)\Delta = \Delta_1$  and  $T(t_2)\Delta_1 = \Delta_2$ , we have

$$\cot \omega_2 = \frac{\{3(t_1 + t_2)^2 + (3t_1 t_2 + 1)^2\} \cos \omega + 6(t_1 + t_2)(3t_1 t_2 + 1)}{2(t_1 + t_2)(3t_1 t_2 + 1) \cos \omega + \{3(t_1 + t_2)^2 + (3t_1 t_2 + 1)^2\}}$$

or, if  $3t_1 t_2 + 1 \neq 0$ ,

$$\cos \omega_2 = \frac{(3t_{12}^2 + 1) \cos \omega + 6t_{12}}{2t_{12} \cos \omega + (3t_{12}^2 + 1)}, \quad (2.7)$$

with

$$t_{12} = \frac{t_1 + t_2}{3t_1 t_2 + 1}, \quad (2.8)$$

giving the multiplication rule for two transformations of the set:  $T(t_2) \cdot \{T(t_1) \cdot \Delta\}$  and  $T(t_{12}) \cdot \Delta$  are equibrocardian triangles. The multiplication is commutative. If  $\Delta$  is a given triangle with Brocard angle  $\omega$  there are two transformations  $T$  such that  $\omega_1 = 0$  and hence  $\cot \omega_1 = \infty$ . From (2.5) it follows that they are  $T(t_i)$ ,  $i = 1, 2$ , such that  $t_i$  are the roots of

$$3t^2 + 2t \cot \omega + 1 = 0, \quad (2.9)$$

that is

$$t_i = \frac{1}{3} \{-\cot\omega \pm (\cot^2\omega - 3)^{1/2}\}. \quad (2.10)$$

If on the sides of a triangle we construct isosceles triangles with  $\tan\varphi = t_1$  or  $t_2$  the points  $A_1, B_1, C_1$  are collinear. As  $t_1 t_2 = 1/3$ ,  $t_1 + t_2 = (-2/3)\cot\omega$  we have  $\varphi_1 + \varphi_2 = \omega - \pi/2$ .

3. Properties of the transformation  $T(t)$  may also be found by vector algebra. We denote the point  $P$  of the plane by the vector  $\bar{P}$  from the origin  $O$  to  $P$ ; the unit vector perpendicular to the plane by  $\bar{e}$ .

We obtain

$$\begin{aligned} 2\bar{A}_1 &= (\bar{B} + \bar{C}) + t\bar{e} \times (\bar{B} - \bar{C}), & 2\bar{B}_1 &= (\bar{C} + \bar{A}) + t\bar{e} \times (\bar{C} - \bar{A}), \\ 2\bar{C}_1 &= (\bar{A} + \bar{B}) + t\bar{e} \times (\bar{A} - \bar{B}) \end{aligned} \quad (3.1)$$

As  $\bar{A}_1 + \bar{B}_1 + \bar{C}_1 = \bar{A} + \bar{B} + \bar{C}$ , we conclude: *the triangles  $\Delta$  and  $T \cdot \Delta$  have the same centroid.*

Furthermore if  $\Delta_1 = T(t_1) \cdot \Delta$  and  $\Delta_2 = T(t_2) \cdot \Delta_1$  we have

$$2\bar{A}_2 = (\bar{B}_1 + \bar{C}_1) + t_2 \bar{e} \times (\bar{B}_1 - \bar{C}_1)$$

or

$$4\bar{A}_2 = (2\bar{A} + \bar{B} + \bar{C}) + (t_1 + t_2) \bar{e} \times (\bar{C} - \bar{B}) + t_1 t_2 (2\bar{A} - \bar{B} - \bar{C}), \quad (3.2)$$

and analogously for  $4\bar{B}_2$  and  $4\bar{C}_2$ . If we take the origin  $O$  at the centroid  $G$  of  $ABC$  we obtain

$$4\bar{A}_2 = (1 + 3t_1 t_2) \bar{A} + (t_1 + t_2) \bar{e} \times (\bar{C} - \bar{B}). \quad (3.3)$$

Two special cases are of interest. If  $t_1 + t_2 = 0$ ,  $-t_2 = +t_1 = t$  one has

$$4\bar{A}_2 = (1 - 3t^2) \bar{A}, \quad 4\bar{B}_2 = (1 - 3t^2) \bar{B}, \quad 4\bar{C}_2 = (1 - 3t^2) \bar{C}. \quad (3.4)$$

Therefore: *If on the sides of  $ABC$  we describe outward (inward) isosceles triangles with base angle  $\varphi$  and then on those of  $A_1 B_1 C_1$  inward (outward) isosceles triangles with the same angle  $\varphi$  then  $A_2 B_2 C_2$  and  $ABC$  are homothetic with respect to the centroid  $G$ , with the scale factor  $(1/4)(1 - 3\tan^2\varphi)$ . The two triangles are congruent if  $\tan\varphi = \pm\sqrt{5/3}$ .*

If  $1 + 3t_1 t_2 = 0$ ,  $t_1 + t_2 = t$  we obtain

$$4\bar{A}_2 = t\bar{e} \times (\bar{C} - \bar{B}), \quad 4\bar{B}_2 = t\bar{e} \times (\bar{A} - \bar{C}), \quad 4\bar{C}_2 = t\bar{e} \times (\bar{B} - \bar{A}). \quad (3.5)$$

Hence the medians of  $A_2 B_2 C_2$  are perpendicular to the sides of  $ABC$  and equal to  $(3/8)ta$ ,  $(3/8)tb$ ,  $(3/8)tc$ ; from this it follows that the sides of  $A_2 B_2 C_2$  are proportional to the medians of  $ABC$  with ratio  $t/2$ .

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## REFERENCES

- [1] L. TOSCANO, *Sur deux suites récurrentes du second ordre, linéaires et homogènes*, Математически Вестник, 9(24), СТР. 131–140 (1972).
- [2] Cf. T. LALESCO, *La géométrie du triangle*, (Paris, 1937), 48–55, 117–120; R. A. JOHNSON, *Advanced Euclidean Geometry* (Dover Publ. New York, 1960), Ch. XVI, The Brocard configuration, 263–286, Ch. XVII, Equibrocardal triangles, 287–301.