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Groups and Fields in Z_n

It can easily be verified that $\{2, 4, 6, 8\}$ is a group under multiplication mod 10 with 6 as the identity and that $\{0, 2, 4, 6, 8\}$ is a field under addition mod 10 and multiplication mod 10. The purpose of this paper is to characterize the subsets of Z_n which are groups under multiplication mod n and those which are fields under addition and multiplication mod n . Some of the results on subgroups of Z_n given here are equivalent to some of the results given by Hewitt and Zuckermann [1], but are of a substantially different form.

In the following $U_n = \{m \mid (n, m) = 1\}$ will denote the group of units in Z_n , ϕ will denote the Euler phi function and a small Roman letter will denote an integer or the residue class of the integer in Z_n ; the context will indicate which is intended.

Proposition 1. If $n = ab$ and $(a, b) = 1$, then $a^{\phi(b)}$ is idempotent in Z_n .

Proof: $a^{\phi(b)} \equiv 1 \pmod{b}$ by Euler's theorem. Hence, multiplying through by $a^{\phi(b)}$ we have $(a^{\phi(b)})^2 \equiv a^{\phi(b)} \pmod{n}$ since $n = ab$ and $(a, b) = 1$. Therefore, $(a^{\phi(b)})^2 = a^{\phi(b)}$ in Z_n .

Lemma 1. If $(x, n) = d$ and $(d, n/d) = 1$, then $x \equiv du \pmod{n}$ where $u \in U_n$. Hence, $x \in dU_n$.

Proof: Since $(d, n/d) = 1$, there exists a t_0 such that $x/d + t_0 n/d \equiv 1 \pmod{d}$. Let $u = x/d + t_0 n/d$. Then $(u, d) = (1, d) = 1$ and $(u, n/d) = (x/d, n/d) = 1$. Hence, $(u, n) = 1$ and $u \in U_n$. Also, $du = x + t_0 n \equiv x \pmod{n}$.

Proposition 2. If $(c, n) = d$ and $(d, n/d) = 1$, then $cU_n = dU_n$.

Proof: By Lemma 1, there is a $u \in U_n$ such that $c \equiv du \pmod{n}$. Hence, $cU_n = duU_n = dU_n$.

Proposition 3. If $n = ab$ and $(a, b) = 1$, then $a^{\phi(b)}U_n = aU_n$.

Proof: This follows immediately from Proposition 2 by observing $(a^{\phi(b)}, n) = a$ and $(a, n/a) = (a, b) = 1$.

The following theorem gives a method for constructing subsets of Z_n which are groups. The example in the first paragraph is obtained by taking $n = 10$ and $a = 2$.

Theorem 1. If $n = ab$ and $(a, b) = 1$, then aU_n is a group under multiplication mod n with $a^{\phi(b)}$ as the identity. The order of this group is $\phi(b)$.

Proof: We will repeatedly make use of the fact $aU_n = a^{\phi(b)}U_n$ obtained in Proposition 3. Let $x, y \in U_n$. Then $(a^{\phi(b)}x)(a^{\phi(b)}y) = (a^{\phi(b)})^2xy = a^{\phi(b)}xy$ by Proposition 1 and $xy \in U_n$. Hence, aU_n is closed under multiplication mod n . Also, $a^{\phi(b)}(a^{\phi(b)}x) = (a^{\phi(b)})^2x = a^{\phi(b)}x$ by Proposition 1. Hence, $a^{\phi(b)}$ is the identity for aU_n . Now let z be the inverse of x in U_n . Then $(a^{\phi(b)}x)(a^{\phi(b)}z) = (a^{\phi(b)})^2xz = a^{\phi(b)}$. Hence, $a^{\phi(b)}z$ is the inverse of $a^{\phi(b)}x$ in aU_n . Therefore, aU_n is a group under multiplication mod n . Furthermore, $ax \equiv ay \pmod{n}$ if and only if $x \equiv y \pmod{b}$ since $(a, n) = a$. Also, if $(w, b) = 1$, then $(aw, n) = a$ and $(a, n/a) = 1$. Hence, by Lemma 1, $aw \in aU_n$. Therefore, there is a one-to-one correspondence between the elements of Z_b which are relatively prime to b and the elements of aU_n . Hence, aU_n has $\phi(b)$ elements.

Theorem 2. Let $(c, n) = d$. cU_n is a group if and only if $(d, n/d) = 1$. In this case $cU_n = dU_n$.

Proof: If $(d, n/d) = 1$, then $cU_n = dU_n$ by Proposition 2 and, by Theorem 1, dU_n is a group. Conversely, if cU_n is a group, then it has an identity of the form cec where $e \in U_n$. Then $cec = c$ in Z_n . Hence, $c^2e \equiv c \pmod{n}$. Therefore, $ce \equiv 1 \pmod{n/d}$. Consequently c is a unit in $Z_{n/d}$ and hence, $(c, n/d) = 1$. Therefore, $(d, n/d) = 1$ since d is a factor of c .

The next theorem shows that all subsets of Z_n which are groups under multiplication mod n are subgroups of the groups given in Theorem 1 and hence the groups in Theorem 1 are maximal. These groups can indeed have proper subgroups which are not obtained by the method in Theorem 1 as is seen by considering the subgroup $\{4, 6\}$ of the group in the example of the first paragraph.

Theorem 3. Let G be a subset of Z_n which is a group under multiplication mod n . If e is the identity for G and $(e, n) = d$, then

- (i) $(x, n) = d$ for every $x \in G$,
- (ii) $(d, n/d) = 1$,
- (iii) G is a subgroup of dU_n .

Proof: Let $x \in G$ with $(x, n) = d'$. Now $ex \equiv x \pmod{n}$. Hence, $e \equiv 1 \pmod{n/d'}$. Since $d \mid e$, $(d, n/d') = 1$ and therefore $d \mid d'$. Also, $x^k \equiv e \pmod{n}$ for some integer $k \geq 2$, if $x \neq e$. Hence, $x^k/d \equiv e/d \pmod{n/d}$. Now $(e/d, n/d) = 1$ and thus $(x^k/d, n/d) = 1$. But d' is a factor of x^k/d since $k \geq 2$. Hence, $(d', n/d) = 1$ and therefore $d' \mid d$. Thus $d = d'$ and $(d, n/d) = 1$. Now, using Lemma 1, we see that if $x \in G$, then $x \in dU_n$; i.e. $G \subseteq dU_n$. Also, by Theorem 2, dU_n is a group and hence G is a subgroup of dU_n .

As an interesting side result we can now show that every idempotent element of Z_n is of the form given in Proposition 1. This characterization of the idempotents of Z_n is quite different from that given in [1].

Corollary 1. Every idempotent element of Z_n is of the form $a^{\phi(b)}$ where $n = ab$ and $(a, b) = 1$.

Proof: If x is an idempotent element of Z_n , then $\{x\}$ is a subset of Z_n which is a group under multiplication mod n . Hence, by Theorem 3 there is an a such that $a \mid n$, $(a, n/a) = 1$ and $\{x\}$ is a subgroup of aU_n . Then x must be the identity of aU_n . Letting $b = n/a$, Theorem 1 tells us $a^{\phi(b)}$ is the identity of aU_n . Hence, $x = a^{\phi(b)}$ in Z_n .

We now turn to the problem of determining the subsets of Z_n which are fields. In the following \overline{U}_n will denote $U_n \cup \{0\}$ and if F is a field, F^* will denote the non-zero elements of F .

Theorem 4. If F is a subset of Z_n which is a field, then there is an a such that $a \mid n$, $(a, n/a) = 1$ and $F = a \overline{U}_n$.

Proof: If F is a subset of Z_n which is a field, then F^* is a group under multiplication mod n . Hence, by Theorem 3 there is an a such that $a \mid n$, $(a, n/a) = 1$ and $F^* \subseteq a U_n$ which is equivalent to $F \subseteq a \overline{U}_n$. Now let e be the multiplicative identity of F , and hence of $a U_n$. Then $a = a e = e + e + \dots + e$ (a summands) must be in F . Also, if $u \in U_n$, $u a = a + a + \dots + a$ (u summands) must be in F . Consequently $a U_n \subseteq F$. Also, $0 \in F$ and thus $F = a \overline{U}_n$.

Theorem 5. Assume $n = ab$ where $(a, b) = 1$. $a \overline{U}_n$ is a field if and only if b is a prime.

Proof: If b is not a prime then $b = cd$ where $c, d > 1$. Now $(ca, n) = ca$ and hence $ca \notin a U_n$ by Theorem 3i. Also $ca \not\equiv 0 \pmod{n}$. Hence, $ca \notin a \overline{U}_n$. But adding a to itself c times gives ca . Therefore, $a \overline{U}_n$ is not closed under addition mod n and hence is not a field. Now assume b is a prime. First of all we observe that $0a, 1a, 2a, \dots, (b-1)a$ are b distinct elements mod n . Also, since $(ia, n) = a(i, b) = a$ for $i = 1, 2, \dots, b-1$ and $(a, n/a) = 1$, $ia \in a U_n$ for $i = 1, 2, \dots, b-1$ by Lemma 1. Since the number of elements in $a U_n$ is $\phi(b) = b-1$, we have $a U_n = \{1a, 2a, \dots, (b-1)a\}$. Hence, $a \overline{U}_n = \{0a, 1a, 2a, \dots, (b-1)a\}$ and this set clearly forms a group under addition mod n . Hence, $a \overline{U}_n$ is a field.

Combining the last two theorems we have the following characterization of the subsets of Z_n which are fields.

Corollary 2. Z_n has subsets which are fields if and only if there exists a prime p such that $p \mid n$ and $p^2 \nmid n$. Moreover, for every such prime p , the set $(n/p) \overline{U}_n$ is a field and all subsets of Z_n which are fields are obtained in this way.

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Kleine Mitteilungen

Ein elementarer Beweis für die Integraldarstellung der Laplaceschen Zahlen

In der numerischen Analysis haben die Laplaceschen Zahlen L_1, L_2, L_3, \dots , neben den Eulerschen und Bernoullischen Zahlen eine grosse Bedeutung erlangt [1]. Sie werden üblicherweise durch die Koeffizienten der Taylor-Reihe

$$-\frac{x}{\ln(1-x)} = 1 - L_1 x - L_2 x^2 - L_3 x^3 - \dots, \quad x \in (-1, 1), \quad (1)$$