

# Groups and fields in Zn

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## Groups and Fields in $Z_n$

It can easily be verified that  $\{2, 4, 6, 8\}$  is a group under multiplication mod 10 with 6 as the identity and that  $\{0, 2, 4, 6, 8\}$  is a field under addition mod 10 and multiplication mod 10. The purpose of this paper is to characterize the subsets of  $Z_n$  which are groups under multiplication mod  $n$  and those which are fields under addition and multiplication mod  $n$ . Some of the results on subgroups of  $Z_n$  given here are equivalent to some of the results given by Hewitt and Zuckermann [1], but are of a substantially different form.

In the following  $U_n = \{m \mid (n, m) = 1\}$  will denote the group of units in  $Z_n$ ,  $\phi$  will denote the Euler phi function and a small Roman letter will denote an integer or the residue class of the integer in  $Z_n$ ; the context will indicate which is intended.

**Proposition 1.** If  $n = ab$  and  $(a, b) = 1$ , then  $a^{\phi(b)}$  is idempotent in  $Z_n$ .

*Proof:*  $a^{\phi(b)} \equiv 1 \pmod{b}$  by Euler's theorem. Hence, multiplying through by  $a^{\phi(b)}$  we have  $(a^{\phi(b)})^2 \equiv a^{\phi(b)} \pmod{n}$  since  $n = ab$  and  $(a, b) = 1$ . Therefore,  $(a^{\phi(b)})^2 = a^{\phi(b)}$  in  $Z_n$ .

**Lemma 1.** If  $(x, n) = d$  and  $(d, n/d) = 1$ , then  $x \equiv du \pmod{n}$  where  $u \in U_n$ . Hence,  $x \in dU_n$ .

*Proof:* Since  $(d, n/d) = 1$ , there exists a  $t_0$  such that  $x/d + t_0 n/d \equiv 1 \pmod{d}$ . Let  $u = x/d + t_0 n/d$ . Then  $(u, d) = (1, d) = 1$  and  $(u, n/d) = (x/d, n/d) = 1$ . Hence,  $(u, n) = 1$  and  $u \in U_n$ . Also,  $du = x + t_0 n \equiv x \pmod{n}$ .

**Proposition 2.** If  $(c, n) = d$  and  $(d, n/d) = 1$ , then  $cU_n = dU_n$ .

*Proof:* By Lemma 1, there is a  $u \in U_n$  such that  $c \equiv du \pmod{n}$ . Hence,  $cU_n = duU_n = dU_n$ .

**Proposition 3.** If  $n = ab$  and  $(a, b) = 1$ , then  $a^{\phi(b)}U_n = aU_n$ .

*Proof:* This follows immediately from Proposition 2 by observing  $(a^{\phi(b)}, n) = a$  and  $(a, n/a) = (a, b) = 1$ .

The following theorem gives a method for constructing subsets of  $Z_n$  which are groups. The example in the first paragraph is obtained by taking  $n = 10$  and  $a = 2$ .

**Theorem 1.** If  $n = ab$  and  $(a, b) = 1$ , then  $aU_n$  is a group under multiplication mod  $n$  with  $a^{\phi(b)}$  as the identity. The order of this group is  $\phi(b)$ .

*Proof:* We will repeatedly make use of the fact  $aU_n = a^{\phi(b)}U_n$  obtained in Proposition 3. Let  $x, y \in U_n$ . Then  $(a^{\phi(b)}x)(a^{\phi(b)}y) = (a^{\phi(b)})^2xy = a^{\phi(b)}xy$  by Proposition 1 and  $xy \in U_n$ . Hence,  $aU_n$  is closed under multiplication mod  $n$ . Also,  $a^{\phi(b)}(a^{\phi(b)}x) = (a^{\phi(b)})^2x = a^{\phi(b)}x$  by Proposition 1. Hence,  $a^{\phi(b)}$  is the identity for  $aU_n$ . Now let  $z$  be the inverse of  $x$  in  $U_n$ . Then  $(a^{\phi(b)}x)(a^{\phi(b)}z) = (a^{\phi(b)})^2xz = a^{\phi(b)}$ . Hence,  $a^{\phi(b)}z$  is the inverse of  $a^{\phi(b)}x$  in  $aU_n$ . Therefore,  $aU_n$  is a group under multiplication mod  $n$ . Furthermore,  $ax \equiv ay \pmod{n}$  if and only if  $x \equiv y \pmod{b}$  since  $(a, n) = a$ . Also, if  $(w, b) = 1$ , then  $(aw, n) = a$  and  $(a, n/a) = 1$ . Hence, by Lemma 1,  $aw \in aU_n$ . Therefore, there is a one-to-one correspondence between the elements of  $Z_b$  which are relatively prime to  $b$  and the elements of  $aU_n$ . Hence,  $aU_n$  has  $\phi(b)$  elements.

**Theorem 2.** Let  $(c, n) = d$ .  $cU_n$  is a group if and only if  $(d, n/d) = 1$ . In this case  $cU_n = dU_n$ .

*Proof:* If  $(d, n/d) = 1$ , then  $cU_n = dU_n$  by Proposition 2 and, by Theorem 1,  $dU_n$  is a group. Conversely, if  $cU_n$  is a group, then it has an identity of the form  $cec$  where  $e \in U_n$ . Then  $cec = c$  in  $Z_n$ . Hence,  $c^2e \equiv c \pmod{n}$ . Therefore,  $ce \equiv 1 \pmod{n/d}$ . Consequently  $c$  is a unit in  $Z_{n/d}$  and hence,  $(c, n/d) = 1$ . Therefore,  $(d, n/d) = 1$  since  $d$  is a factor of  $c$ .

The next theorem shows that all subsets of  $Z_n$  which are groups under multiplication mod  $n$  are subgroups of the groups given in Theorem 1 and hence the groups in Theorem 1 are maximal. These groups can indeed have proper subgroups which are not obtained by the method in Theorem 1 as is seen by considering the subgroup  $\{4, 6\}$  of the group in the example of the first paragraph.

**Theorem 3.** Let  $G$  be a subset of  $Z_n$  which is a group under multiplication mod  $n$ . If  $e$  is the identity for  $G$  and  $(e, n) = d$ , then

- (i)  $(x, n) = d$  for every  $x \in G$ ,
- (ii)  $(d, n/d) = 1$ ,
- (iii)  $G$  is a subgroup of  $dU_n$ .

*Proof:* Let  $x \in G$  with  $(x, n) = d'$ . Now  $ex \equiv x \pmod{n}$ . Hence,  $e \equiv 1 \pmod{n/d'}$ . Since  $d \mid e$ ,  $(d, n/d') = 1$  and therefore  $d \mid d'$ . Also,  $x^k \equiv e \pmod{n}$  for some integer  $k \geq 2$ , if  $x \neq e$ . Hence,  $x^k/d \equiv e/d \pmod{n/d}$ . Now  $(e/d, n/d) = 1$  and thus  $(x^k/d, n/d) = 1$ . But  $d'$  is a factor of  $x^k/d$  since  $k \geq 2$ . Hence,  $(d', n/d) = 1$  and therefore  $d' \mid d$ . Thus  $d = d'$  and  $(d, n/d) = 1$ . Now, using Lemma 1, we see that if  $x \in G$ , then  $x \in dU_n$ ; i.e.  $G \subseteq dU_n$ . Also, by Theorem 2,  $dU_n$  is a group and hence  $G$  is a subgroup of  $dU_n$ .

As an interesting side result we can now show that every idempotent element of  $Z_n$  is of the form given in Proposition 1. This characterization of the idempotents of  $Z_n$  is quite different from that given in [1].

**Corollary 1.** Every idempotent element of  $Z_n$  is of the form  $a^{\phi(b)}$  where  $n = ab$  and  $(a, b) = 1$ .

*Proof:* If  $x$  is an idempotent element of  $Z_n$ , then  $\{x\}$  is a subset of  $Z_n$  which is a group under multiplication mod  $n$ . Hence, by Theorem 3 there is an  $a$  such that  $a \mid n$ ,  $(a, n/a) = 1$  and  $\{x\}$  is a subgroup of  $aU_n$ . Then  $x$  must be the identity of  $aU_n$ . Letting  $b = n/a$ , Theorem 1 tells us  $a^{\phi(b)}$  is the identity of  $aU_n$ . Hence,  $x = a^{\phi(b)}$  in  $Z_n$ .

We now turn to the problem of determining the subsets of  $Z_n$  which are fields. In the following  $\overline{U}_n$  will denote  $U_n \cup \{0\}$  and if  $F$  is a field,  $F^*$  will denote the non-zero elements of  $F$ .

**Theorem 4.** If  $F$  is a subset of  $Z_n$  which is a field, then there is an  $a$  such that  $a \mid n$ ,  $(a, n/a) = 1$  and  $F = a \overline{U}_n$ .

*Proof:* If  $F$  is a subset of  $Z_n$  which is a field, then  $F^*$  is a group under multiplication mod  $n$ . Hence, by Theorem 3 there is an  $a$  such that  $a \mid n$ ,  $(a, n/a) = 1$  and  $F^* \subseteq a U_n$  which is equivalent to  $F \subseteq a \overline{U}_n$ . Now let  $e$  be the multiplicative identity of  $F$ , and hence of  $a U_n$ . Then  $a = a e = e + e + \dots + e$  ( $a$  summands) must be in  $F$ . Also, if  $u \in U_n$ ,  $u a = a + a + \dots + a$  ( $u$  summands) must be in  $F$ . Consequently  $a U_n \subseteq F$ . Also,  $0 \in F$  and thus  $F = a \overline{U}_n$ .

**Theorem 5.** Assume  $n = ab$  where  $(a, b) = 1$ .  $a \overline{U}_n$  is a field if and only if  $b$  is a prime.

*Proof:* If  $b$  is not a prime then  $b = cd$  where  $c, d > 1$ . Now  $(ca, n) = ca$  and hence  $ca \notin a U_n$  by Theorem 3i. Also  $ca \not\equiv 0 \pmod{n}$ . Hence,  $ca \notin a \overline{U}_n$ . But adding  $a$  to itself  $c$  times gives  $ca$ . Therefore,  $a \overline{U}_n$  is not closed under addition mod  $n$  and hence is not a field. Now assume  $b$  is a prime. First of all we observe that  $0a, 1a, 2a, \dots, (b-1)a$  are  $b$  distinct elements mod  $n$ . Also, since  $(ia, n) = a(i, b) = a$  for  $i = 1, 2, \dots, b-1$  and  $(a, n/a) = 1$ ,  $ia \in a U_n$  for  $i = 1, 2, \dots, b-1$  by Lemma 1. Since the number of elements in  $a U_n$  is  $\phi(b) = b-1$ , we have  $a U_n = \{1a, 2a, \dots, (b-1)a\}$ . Hence,  $a \overline{U}_n = \{0a, 1a, 2a, \dots, (b-1)a\}$  and this set clearly forms a group under addition mod  $n$ . Hence,  $a \overline{U}_n$  is a field.

Combining the last two theorems we have the following characterization of the subsets of  $Z_n$  which are fields.

Corollary 2.  $Z_n$  has subsets which are fields if and only if there exists a prime  $p$  such that  $p \mid n$  and  $p^2 \nmid n$ . Moreover, for every such prime  $p$ , the set  $(n/p) \overline{U}_n$  is a field and all subsets of  $Z_n$  which are fields are obtained in this way.

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## Kleine Mitteilungen

### Ein elementarer Beweis für die Integraldarstellung der Laplaceschen Zahlen

In der numerischen Analysis haben die Laplaceschen Zahlen  $L_1, L_2, L_3, \dots$ , neben den Eulerschen und Bernoullischen Zahlen eine grosse Bedeutung erlangt [1]. Sie werden üblicherweise durch die Koeffizienten der Taylor-Reihe

$$-\frac{x}{\ln(1-x)} = 1 - L_1 x - L_2 x^2 - L_3 x^3 - \dots, \quad x \in (-1, 1), \quad (1)$$