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gelte auch für den Fall, dass die Scheibe durch einen Massenpunkt ersetzt wird. Dies ist indes nicht richtig. Der Grenzübergang $r \to 0$ ist nicht erlaubt, weil bei ihm bei D nach (3) (r/ϱ ginge gegen 1) schon in 0. Ordnung ein unendlich grosser Koeffizient von t^2 entstünde. Man kann auch sagen: Der Konvergenzbereich der Methode in t würde für $r \to 0$ auf Null zusammenschrumpfen. Man kann jedoch das Ergebnis für den Massenpunkt durch einen einfachen Kunstgriff erhalten: Man unterbindet die für den Massenpunkt sinnlose Drehung durch ein bei verschwindendem r endlich bleibendes Trägheitsmoment, also durch $\alpha = 0$. Dies in (12) eingesetzt, liefert sofort für den Massenpunkt:

$$R_{M.P.} = -6 \frac{f}{s_4 u} (f - u) . \tag{14}$$

6. *Der Fall der starren Saiten*

Sind die Aufhängesaiten nicht dehnbar, so ist $f = u =$ const. Das formale Ergebnis nach (12) oder (13) wäre $R = 0$, was sicher unrichtig ist. Tatsächlich ist auch dieser Grenzübergang nicht erlaubt, da er durch $k \to \infty$ auszudrücken wäre, wobei in (3) wieder schon in 0. Ordnung die Koeffizienten von t^2 unendlich würden. Während eben bei elastischen Saiten beim Zerschneiden der rechten Saite die Spannkraft der linken sich nur allmählich, gemäss ihrer Längenänderung, ändert, wird sie im Fall der starren Saite, die keine Längenänderung zulässt, schlagartig anders, in ihrem Betrag erzwungen durch die Zwangsbedingung $f =$ const. Es gibt hier nicht wie im Fall des Massenpunktes einen Kunstgriff, um doch noch ein richtiges Ergebnis zu gewinnen. Die ganze Rechnung müsste vielmehr von vornherein mit Hilfe der Zwangsbedingung anders geführt werden.

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The Cubic Revisited

In this note we offer another *method* for solving the general cubic equation. Our method amounts to the technique of completing the cube in a transformed equation by which we arrive at a binomial equation. Our *formula* is of course not new in every way, and naturally it can be reconciled with the celebrated one by Cardan (Tartaglia).

The general cubic equation

$$f(x) = a\,x^3 + b\,x^2 + c\,x + d = 0\,, \quad (a \neq 0)$$

with coefficients in the field of complex numbers may be written in the form

$$g(y) = y^3 + q\,y + r = 0$$

by means of the transformation $x = y - b/3a$. The roots of $f(x) = 0$ are therefore uniquely determined by those of $g(y) = 0$. We consider this reduced equation.

We shall say that the cube can be completed in

$$p(x) = p_0\,x^3 + p_1\,x^2 + p_2\,x + p_3, \quad (p_i \in C,\ p_0 \neq 0)\,,$$

if $p(x)$ can be written in the form

$$p(x) = (\alpha\,x + \beta)^3 + \gamma$$

where α is any fixed cube root of p_0 and β and γ are complex numbers. It is a matter of routine to show that the cube can be completed in $p(x)$ if and only if $p_1^2 = 3\,p_0 p_2$. In this case we have

$$p(x) = \left(\alpha\,x + \frac{p_1}{3\,\alpha^2}\right)^3 + p_3 - \frac{p_1\,p_2}{9\,p_0}\,.$$

These facts will be used to determine the roots of $g(y) = 0$. We consider two possibilities.

Case 1. The cube can be completed in $g(y)$. This condition holds if and only if $3\,q = 0$ or equivalently $q = 0$. In this case the equation $g(y) = 0$ has the binomial form $y^3 = -r$ and the roots are given by $y = \delta_i$, $(i = 0, 1, 2)$ where δ_i are the three cube roots of $-r$. If in addition $r = 0$, the equation has the repeated root 0 of multiplicity three.

Remark. The condition $q = r = 0$ offers the only possibility of a repeated root of multiplicity three, a fact which can easily be verified by considering $g(y) = g'(y) = g''(y) = 0$. Furthermore, the condition $q = 0$ rules out the possibility of a repeated root of multiplicity two, for obviously, $g(y) = g'(y) = 0$ and $g''(y) \neq 0$ cannot hold simultaneously in this case.

Case 2. The cube cannot be completed in $g(y)$, i.e. $q \neq 0$. In this case we transform $g(y) = 0$ into an equation either with one root isolated immediately or with a left hand side in which the cube can be completed. This is done in terms of a parameter, the values of which are to be determined. The transformation is $y = z^{-1} + k$ and the resulting equation is

$$h(z) = g(k)\,z^3 + g'(k)\,z^2 + (1/2)\,g''(k)\,z + 1 = 0\,. \tag{1}$$

We now consider our necessary and sufficient condition for completing the cube in $h(z)$, namely

$$[g'(k)]^2 = \frac{3}{2}\,g(k)\,g''(k)\,. \tag{2}$$

This condition is directly shown to be equivalent to $3qk^2 + 9rk - q^2 = 0$ which is the case if and only if k has any of the values

$$k_1, k_2 = \frac{-9r \pm \sqrt{81r^2 + 12q^3}}{6q} = \frac{-9r \pm s}{6q}$$

where we take $s = \sqrt{81r^2 + 12q^3}$ as the root of $s^2 = 81r^2 + 12q^3$ with $\mathrm{Re}(s) \geq 0$ ($\mathrm{Im}(s) > 0$) if the roots are not (are) purely imaginary.

If $g(k) = 0$ for any (and hence both) of these values of k, then $h(z)$ is not cubic, but we have the solution! If $g(k_i) \neq 0$, then $h(z)$ is cubic and we may complete the cube in (1) to solve for z and hence for y. We consider these two possibilities separately.

(i) $g(k_i) = 0$. By (2) we have that $g'(k_i) = 0$, i.e. $3k_i^2 + q = 0$. Since $q \neq 0$ it follows that $k_i \neq 0$ so that $g''(k_i) = 6k_i \neq 0$. Hence k_i, $(i = 1, 2)$, is a repeated root of multiplicity two. (The latter conclusion holds if and only if $k_1 = k_2$, i.e. if and only if $s = 0$ or equivalently the well known $\Delta = (r/2)^2 + (q/3)^3 = 0$.) Conversely, if k is a repeated root of multiplicity two then $g(k) = g'(k) = 0$, so that (2) holds and hence $k = k_1 = k_2$. Thus $g(k_i) = 0$ is a necessary and sufficient condition for a repeated root of multiplicity two. The repeated root in this case is given by $y = -9r/6q = -3r/2q$ and the remaining root follows easily from the fact that the sum of the three roots vanishes.

(ii) $g(k_i) \neq 0$. In view of what has been said in (i) it is clear that the present condition is a necessary and sufficient one for three mutually distinct roots in the case where $q \neq 0$. We note also that $s \neq 0$ here. Now using the value k_1 in (1), completing the cube and simplifying, we obtain the equation

$$\left[\alpha z + \frac{g'(k_1)}{3\alpha^2}\right]^3 = \frac{[g'(k_1)]^3}{27\alpha^6} - 1$$

where α is a fixed cube root of $g(k_1)$. This implies that

$$\left[z + \frac{g'(k_1)}{3g(k_1)}\right]^3 = \left[\frac{g'(k_1)}{3g(k_1)}\right]^3 - \frac{1}{g(k_1)}.$$

Using the fact that $12q^3 = s^2 - 81r^2$ it easily follows that

$$\left(z + \frac{3q}{s}\right)^3 = \frac{-18(9r+s)^2}{8s^3}$$

which implies that $(2sz + 6q)^3 = -18(9r+s)^2$.

Setting $z = 1/(y - k)$ and simplifying, we obtain the equation

$$\left[\frac{6q(y-k_2)}{y-k_1}\right]^3 = -18(9r+s)^2$$

which reduces to the simple form

$$\left[\frac{y-k_2}{y-k_1}\right]^3 = \frac{k_2}{k_1}.$$

Hence $(y - k_2)/(y - k_1) = \varepsilon_i$, $(i = 0, 1, 2)$, where ε_i are the three cube roots of

$$\frac{k_2}{k_1} = \frac{-\frac{r}{2} - \sqrt{\left(\frac{r}{2}\right)^2 + \left(\frac{q}{3}\right)^3}}{-\frac{r}{2} + \sqrt{\left(\frac{r}{2}\right)^2 + \left(\frac{q}{3}\right)^3}} .$$

Direct solution yields

$$y = \frac{k_2 - \varepsilon_i \, k_1}{1 - \varepsilon_i} ,$$

a formula which is readily simplified to give $y = - k_1 (\varepsilon_i + \varepsilon_i^2)$, or more explicitly,

$$y = \frac{3}{q} \left[\frac{r}{2} - \sqrt{\left(\frac{r}{2}\right)^2 + \left(\frac{q}{3}\right)^3}\right] (\varepsilon_i + \varepsilon_i^2) , \quad (i = 0, 1, 2) .$$

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Kleine Mitteilungen

On a theorem of Cipolla

Cipolla proved 1904 in [1] the following theorem: The number

$$\left(2^{2^m} + 1\right) \left(2^{2^n} + 1\right) \cdots \left(2^{2^s} + 1\right) ,$$

with $m > n > \ldots > s$, is a pseudoprime if and only if $2^s > m$ (a positive integer n is called a pseudoprime if $n \mid 2^n - 2$ and n is composite).

In many applications it is usefull to have 'strong' pseudoprimes. In the following definition we give a precise meaning to this concept:

Definition: The positive integer n is a k-th order pseudoprime if and only if $k \mid n - 1$, $2^{(n-1)/k} \equiv 1 \pmod{n}$ and n is composite.

In this paper we prove the following generalization of Cipolla's result:

Theorem: $L = \left(2^{2^m} + 1\right) \left(2^{2^n} + 1\right) \ldots \left(2^{2^s} + 1\right)$, with $m > n > \cdots > s$, is a 2^t-th order pseudoprime if and only if $2^s > m + t$.