Kleine Mitteilungen

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Refinements of a Congruence of Gauss

1. Let h and k denote relatively prime odd positive integers. Replacing each multiple jh, for $1 \leq j \leq (k-1)/2$, by its least (in absolute value) residue modulo k, let $m_k(h)$ denote the number of these that are negative, and $M_k(h)$ the sum of these negative integers.

It has long been known that the residue classes modulo 4 of h and k determine the parity of $m_k(h)+m_h(k)$. This follows from the congruence

$$m_k(h)+m_h(k)+\frac{h-1}{2}\frac{k-1}{2} \equiv 0 \pmod 2,$$

first demonstrated by Gauss for use in his third proof of the law of quadratic reciprocity.

In fact, the following stronger congruence holds.

Theorem 1.

$$m_k(h)+m_h(k)+\frac{h-1}{2}\frac{k-1}{2} \equiv 0 \pmod 4.$$

From this result, one sees that the residue class modulo 4 of $m_k(h)+m_h(k)$ is determined by the residue classes modulo 8 of h and k. That these latter classes also determine the parity of $M_k(h)+M_h(k)$ follows from:

Theorem 2.

$$M_k(h)+M_h(k) \equiv \frac{h-1}{2}\frac{k^2-1}{8}+\frac{k-1}{2}\frac{h^2-1}{8} \pmod 2.$$

2. The novelty of the above statements is somewhat surprising in that they appear quite simply as the offspring of rather aged parents. This section lists these results, the only innovative feature being the proof of (3) which is considerably simpler than that given in [1], p. 229–230.

The formula

$$m_k(h)=\sum_{j=1}^{(k-1)/2}\left\{\left[\frac{jh}{k}+\frac{1}{2}\right]-\left[\frac{jh}{k}\right]\right\} \tag{1}$$

notes the fact that the contribution of the $j-th$ term is one if the least residue of $jh \pmod k$ is negative and zero otherwise.

The classical reciprocity formula

$$\sum_{i=1}^{(h-1)/2}\left[\frac{ik}{h}\right]+\sum_{j=1}^{(k-1)/2}\left[\frac{jh}{k}\right]=\frac{h-1}{2}\frac{k-1}{2} \tag{2}$$

is proved by counting the ordered pairs (i,j) with $1 \leq i \leq (h-1)/2$, $1 \leq j \leq (k-1)/2$. The first sum counts those pairs for which $jh<ik$, the second those for which $jh>ik$, the equation $jh=ik$ being insoluble.

Now, counting only those pairs for which $(2i-1)k \leq 2jh$, it is easily seen that they number

$$\sum_{j=1}^{(k-1)/2}\left[\frac{jh}{k}+\frac{1}{2}\right].$$

Defining the associated pair (i^*,j^*) by $i^*=\frac{h+1}{2}-i, j^*=\frac{k+1}{2}-j$, one verifies the equivalence of the two inequalities $(2i-1)k \leq 2jh$ and $(2j^*-1)h \leq 2i^*k$. It follows immediately that

$$\sum_{i=1}^{(h-1)/2}\left[\frac{ik}{h}+\frac{1}{2}\right]=\sum_{j=1}^{(k-1)/2}\left[\frac{jh}{k}+\frac{1}{2}\right]. \tag{3}$$

Implicit in [2], p. 65–66, one has

$$\frac{(h-1)(k^2-1)}{8}=k\left(m_k(h)+\sum_{j=1}^{(k-1)/2}\left[\frac{jh}{k}\right]\right)+2M_k(h). \tag{4}$$

Since the left hand side of (4) is even,

$$m_k(h) \equiv \sum_{j=1}^{(k-1)/2}\left[\frac{jh}{k}\right] \pmod 2 \tag{5}$$

and, combining this with (1),

$$\sum_{j=1}^{(k-1)/2}\left[\frac{jh}{k}+\frac{1}{2}\right] \equiv 0 \pmod 2. \tag{6}$$

3. The proofs of the theorems can now be rapidly completed. Combining (3) and (6) gives

$$\sum_{i=1}^{(h-1)/2}\left[\frac{ik}{h}+\frac{1}{2}\right]+\sum_{j=1}^{(k-1)/2}\left[\frac{jh}{k}+\frac{1}{2}\right] \equiv 0 \pmod 4.$$

Inserting this in (1),

$$\left(m_k(h)+\sum_{j=1}^{(k-1)/2}\left[\frac{jh}{k}\right]\right)+\left(m_h(k)+\sum_{i=1}^{(h-1)/2}\left[\frac{ik}{h}\right]\right) \equiv 0 \pmod 4.$$

Theorem 1 now follows from (2). Moreover, by (5), each of the expressions in parentheses is even, and so,

$$k\left(m_k(h)+\sum_{j=1}^{(k-1)/2}\left[\frac{jh}{k}\right]\right)+h\left(m_h(k)+\sum_{i=1}^{(h-1)/2}\left[\frac{ik}{h}\right]\right) \equiv 0 \pmod 4.$$

Theorem 2 follows by virtue of (4).

Remark. It should be noted that a completely different proof of these results may be given by using a result on Dedekind sums due to ROSEN [3].

John B. Friedlander and Kenneth H. Rosen,
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Elementarmathematik und Didaktik

Eine besondere Art gleichseitiger Sechsecke

Bei gleichseitigen Streckenzügen, deren Ecken abwechselnd auf zwei konzentrischen Kreisen liegen, gilt eine einfache Winkelbeziehung. Die Umkehrfrage führt in 1. auf besondere Sechsecke. Sie werden in 2. rein algebraisch gefunden. In 3. wird auf ein besonderes Gelenkviereck hingewiesen.

1. Definition und Eigenschaften der Sechsecke Σ

Gegeben seien in der euklidischen Ebene die Kreise k_0, k_1 (Mitten K_0, K_1, Radien r_0, r_1, Abstand $\overline{K_0K_1} = e$). Von einem Punkt *0* auf k_0 und einem Punkt *1* auf k_1 ausgehend, sei in Abb. 1 der Streckenzug ... $-2, -1, 0, 1, 2, 3,$ ... konstruiert, bei dem Nachbarecken den konstanten Abstand $\overline{v, v+1} = \overline{0, 1} = l$ haben (v ganz). a_v sei der (orientierte) Winkel der (orientierten) Geraden K_0K_1 mit K_0v (wenn v gerade ist) bzw. K_1v (v ungerade). $\bar{1}, \bar{2}$ seien die Fusspunkte der Normalen aus *1* bzw. *2* auf K_0K_1. Wegen $\sphericalangle K_1K_01 = (a_0+a_2)/2$ und $180° - \sphericalangle K_0K_12 = (a_1+a_3)/2$ folgt

$$\text{a) } \operatorname{tg}\frac{a_0+a_2}{2} = \frac{\bar{1}1}{K_0\bar{1}} = \frac{r_1 \sin a_1}{r_1 \cos a_1 + e}, \quad \text{b) } \operatorname{tg}\frac{a_1+a_3}{2} = \frac{\bar{2}2}{K_1\bar{2}} = \frac{r_0 \sin a_2}{r_0 \cos a_2 - e}. \quad \text{(1a, b)}$$

Im Sonderfall $e=0$ ist $\operatorname{tg}\frac{a_0+a_2}{2} = \operatorname{tg}\frac{a_1}{2}, \ldots$ Es folgt $a_v - 2a_{v+1} + a_{v+2} \equiv 0$ (mod 360°).

Wir stellen nun die Umkehrfrage: Besteht eine lineare Beziehung $a_v + a_{v+2} = \lambda a_{v+1} + \mu$ *mit Konstanten* λ, μ *nur bei konzentrischen Kreisen?* Nach (1a) müsste $(a_0 + a_2 =)\ 2 \operatorname{arctg}\,(r_1 \sin a_1/(r_1 \cos a_1 + e)) = \lambda a_1 + \mu$ für alle a_1 gelten. Hieraus folgt durch zweimalige Differentiation nach a_1: Für alle a_1 muss $er_1(r_1^2 - e^2)\sin a_1/(e^2 + 2er_1 \cos a_1 + r_1^2)^2 = 0$ sein. Wegen $r_1 \neq 0$ folgt, dass $e=0$ oder $e=r_1$ sein muss. Aus (1b) folgt ebenso, dass $e=0$ oder $e=r_2$ sein muss. Der Fall