Kleine Mitteilungen

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On the Indirect Product of Subgroups

The following is a well-known property of external direct products[1]). If G is the external direct product of groups $G_1, G_2, \ldots, G_n$, and H is a group such that $H = \bigtimes_{i=1}^{k} G_i$ for some $k < n$, then G is, up to isomorphism, the external direct product of H, $G_{k+1}, \ldots, G_n$.

Recall that a group G is said to be the internal direct product of subgroups H_1, H_2, ..., H_n if and only if each H_i is normal and each element g of G can be uniquely expressed (excluding order) in the form $g = h_1 h_2 \ldots h_n$, where $h_i \in H_i$ for each i.

It is natural to ask the following question: If G is the internal direct product of subgroups $H_1, H_2, \ldots, H_n$, what conditions can be placed on a subgroup H so that G is also the internal direct product of H, $H_{k+1}, \ldots, H_n$ for some $k < n$? I have been unable to find a text on group theory which considers this question.

For convenience let us first recall some basic results concerning direct products. A well-known characterization is that a group G is the internal direct product of subgroups $H_1, H_2, \ldots, H_n$ if and only if

$$\text{each } H_i \text{ is normal}, \tag{1}$$

$$G = \prod_{i=1}^{n} H_i, \text{ and} \tag{2}$$

$$H_j \cap \prod_{i \neq j} H_i = \{e\} \text{ for each } j. \tag{3}$$

Another characterization is that G is the internal direct product if and only if $\Phi\colon \mathop{\times'}_{i=1}^{n} H_i \to G$, defined by $\Phi(h_1, h_2, \ldots, h_n) = h_1 h_2 \ldots h_n$, is an isomorphism.

Returning to the question, one might first guess that if $H \simeq \prod_{i=1}^{k} H_i$ for some $k < n \left(\prod_{i=1}^{k} H_i \text{ is a subgroup since each } H_i \text{ is normal and } H_i \cap \prod_{i \neq j} H_j = \{e\} \right)$, then G will be the internal direct product of $H, H_{k+1}, \ldots, H_n$ since $G \simeq \mathop{\times}_{i=1}^{n} H_i, H \simeq \mathop{\times}_{i=1}^{k} H_i$, and G will thus be isomorphic to the external direct product of $H, H_{k+1}, \ldots, H_n$. However, the special map Φ may not be an isomorphism. Consider $Z_2 \times Z_2$, and let $H_1 = Z_2 \times \{[0]\}$, $H_2 = \{[0]\} \times Z_2$, and $H = H_2$. Then $Z_2 \times Z_2$ is the internal direct product of H_1 and H_2, and $H \simeq H_1$, but even though $Z_2 \times Z_2 \simeq H \times H_2$, $Z_2 \times Z_2$ is not the internal direct product of H and H_2 since $H + H_2 \neq Z_2 \times Z_2$.

This example also shows that requiring H to be normal in G, in addition to

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[1]) External direct product means the same as "ordinary" direct product as indicated by the symbol $\bigtimes_{i=1}^{n} G_i$.

being isomorphic to $\prod_{i=1}^{k} H_i$, does not suffice. But in the example just cited, note that $H \cap H_2 \neq \{([0], [0])\}$. Thus, in addition to supposing H to be isomorphic to $\prod_{i=1}^{k} H_i$, let us assume that $H \cap \prod_{i=k+1}^{n} H_i = \{e\}$. But consider $Z_2 \times S_3$, and let $H_1 = Z_2 \times \{\varepsilon\}$, $H_2 = \{[0]\} \times S_3$, and $H = \{([0], \varepsilon), ([1], a)\}$, where $\varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ and $a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Then $Z_2 \times S_3$ is the internal direct product of H_1 and H_2, $H \simeq H_1$, and $H \cap H_2 = \{([0], \varepsilon)\}$, but $Z_2 \times S_3$ is not the internal direct product of H and H_2 since H is not normal.

If we restrict ourselves to finite groups, we have the following.

Theorem: Let G be a finite group such that G is the internal direct product of subgroups $H_1, H_2, \ldots, H_n$. If H is a normal subgroup such that $H \simeq \prod_{i=1}^{k} H_i$ for some $k < n$ and $H \cap \prod_{i=k+1}^{n} H_i = \{e\}$, then G is the internal direct product of $H, H_{k+1}, \ldots, H_n$.

Proof: It suffices to show that $\Phi\colon H \times H_{k+1} \times \ldots \times H_n \to G$, defined by $\Phi(h, h_{k+1}, \ldots, h_n) = h h_{k+1} \ldots h_n$, is an isomorphism.

If K_1 and K_2 are normal subgroups and $K_1 \cap K_2 = \{e\}$, then the elements of K_1 commute with those of K_2. Using this, the proof that Φ is a homomorphism is straightforward.

If $\Phi(a, a_{k+1}, \ldots, a_n) = \Phi(b, b_{k+1}, \ldots, b_n)$, then $a a_{k+1} \ldots a_n = b b_{k+1} \ldots b_n$. Hence $b^{-1} a = (b_{k+1} \ldots b_n)(a_{k+1} \ldots a_n)^{-1}$. Thus $b^{-1} a \in H \cap \prod_{i=k+1}^{n} H_i$, and $a = b$. Continuing in the same manner, $a_i = b_i$ for $i = k+1, \ldots, n$. Thus Φ is one-to-one. Since $H \simeq \prod_{i=1}^{n} H_i$, the orders of $H \times H_{k+1} \times \ldots \times H_n$ and of G are equal. Thus it follows that Φ is also onto. Therefore, Φ is an isomorphism.

I have answered the question for finite groups only. If G is infinite, the same conditions imposed on H are not sufficient to obtain the same result. Consider $Z \times Z$, and let $H_1 = Z \times \{0\}$, $H_2 = \{0\} \times Z$, and H be the subgroup generated by (2,0). Then $Z \times Z$ is the internal direct product of H_1 and H_2, H is a normal subgroup such that $H \simeq H_1$ and $H \cap H_2 = \{(0,0)\}$, but $Z \times Z$ is not the internal direct product of H and H_2 since $H + H_2 \neq Z \times Z$.

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