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Autor: Zaks, J. / Perles, M.A. / Wills, J.M.
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On lattice polytopes having interior lattice points

Let $V(K)$ denote the volume of a convex body K in the Euclidean d -space E^d , and let $G(K)$ and $G^0(K)$ be defined by $G(K) = \text{card}(K \cap \mathbb{Z}^d)$ and $G^0(K) = \text{card}(\text{int } K \cap \mathbb{Z}^d)$, where $\mathbb{Z}^d$ denotes the set of all the lattice points in E^d , i.e., points having only integral coordinates.

The following theorems of Minkowski are well known.

Theorem A (see [2], p. 76). *If K is a centrally symmetric convex body in E^d and $G^0(K) = 1$, then $V(K) \leq 2^d$.*

Theorem B (see [2], p. 79 and 96). *If K is a centrally symmetric convex body in E^d and $G^0(K) = 1$, then $G(K) \leq 3^d$; if $G^0(K) = 1$ and $V(K) = 2^d$, then K is a convex polytope and it has at most $2^{d+1} - 2$ facets.*

There are no analogues of theorems A and B for convex bodies which are not centrally symmetric. The purpose of this paper is to give some information on $V(P)$ and $G(P)$ in the case where P is a lattice polytope (i.e., P has all of its vertices in $\mathbb{Z}^d$) satisfying $G^0(P) = n, n \geq 1$. We will see that V and G behave very different from the centrally symmetric case.

Let $g(d, n)$ and $v(d, n)$ be defined for $n \geq 1$ and $d \geq 2$ as follows: $g(d, n) = \sup \{G(P) \mid P \subset E^d, G^0(P) = n\}$ and $v(d, n) = \sup \{V(P) \mid P \subset E^d, G^0(P) = n\}$.

Scott, [3] proved that $g(2, 1) = 10$ and that $g(2, n) = 3n + 6$ for all $n \geq 2$. Our main result is the following

Theorem. *For all $d, d \geq 4$, and for all $n, n \geq 1$,*

$$v(d, n) \geq \frac{n+1}{d!} 2^{2^{d-a}} \quad \text{and} \quad g(d, n) \geq \frac{n+1}{6(d-2)!} 2^{2^{d-a}},$$

where $a = 0.5856 \dots$; $v(3, n) \geq 6(n+1)$, $g(3, n) \geq 16n + 23$, $v(4, 1) \geq 147$ and $g(4, 1) \geq 680$.

We need the following

Lemma. *If $(a_n)_{n=1}^\infty$ is defined by $a_1 = 2$ and $a_d = \prod_{i=1}^{d-1} a_i + 1$ for $d \geq 2$, then*

(i) $a_1 = 2, a_2 = 3, a_3 = 7, a_4 = 43$ and $a_5 = 1807$;

(ii) for all $d, d \geq 1, 1 - \sum_{i=1}^d \frac{1}{a_i} = \left(\prod_{i=1}^d a_i \right)^{-1}$;

(iii) for all $d, d \geq 4, \prod_{i=1}^d a_i \geq 2^{2^{d-a}}$ where $a = 0.5856 \dots$

Proof of the lemma: (i) is trivial, (ii) is true for $d=1$, and if it is true for d , then by induction

$$1 - \sum_{i=1}^{d+1} \frac{1}{a_i} = \left(\prod_{i=1}^d a_i \right)^{-1} - \frac{1}{a_{d+1}} = \left(\prod_{i=1}^{d+1} a_i \right)^{-1}.$$

Let the real a be defined by $2^{2^{d-a}} = a_1 a_2 a_3 a_4 = 1806$, with $d=4$; thus $a=0.5856 \dots$. Suppose (iii) holds for d , then

$$\prod_{i=1}^{d+1} a_i > \left(\prod_{i=1}^d a_i \right)^2 \geq 2^{2 \cdot 2^{d-a}} = 2^{2^{d+1-a}},$$

hence it holds for $d+1$.

Proof of the theorem: Let $k_i = a_i$ for $i=1, \dots, d-1$, $k_d = 2(a_d - 1)$, and let S_1^d be the d -simplex defined by

$$S_1^d = \left\{ (x_1, \dots, x_d) \in E^d \mid x_i \geq 0, \sum_{i=1}^d \frac{x_i}{k_i} \leq 1 \right\}.$$

By (ii) we have

$$\sum_{i=1}^d \frac{1}{k_i} < 1 \quad \text{and} \quad \frac{1}{k_d} + \sum_{i=1}^d \frac{1}{k_i} = 1,$$

hence $(1, \dots, 1) \in \text{int } S_1^d$ and $(1, \dots, 1, 2) \notin \text{int } S_1^d$; $k_1 < k_2 < \dots < k_d$ imply that $G^0(S_1^d) = 1$. For general n , $n \geq 1$, replace $k_d = 2(a_d - 1)$ by $k_d = (n+1)(a_d - 1)$, and obtain the d -simplex S_n^d . It follows that the only lattice points of $\text{int } S_n^d$ are $(1, \dots, 1, j)$ for $1 \leq j \leq n$, hence $G^0(S_n^d) = n$.

As for the volume of S_n^d , $V(S_n^d) = (1/d!) \prod_{i=1}^d k_i = ((n+1)/d!) (a_d - 1)^2$ and by (iii):

$$a_d - 1 \geq 2^{2^{d-1-a}}, \quad \text{therefore} \quad v(d, n) \geq \frac{n+1}{d!} 2^{2^{d-a}}.$$

To compute $G(S_n^d)$ let $d \geq 3$ and $S = S_n^d \cap \{x_1 = 0\} \cap \{x_2 = 0\}$.

It is easy to see that for each $t \in \text{aff } S$

$G(S+t) \leq G(S)$ and equality iff x is a lattice point.

If C_{d-2} denotes the unit cube in $\text{aff } S$ and V_{d-2} the volume in $\text{aff } S$, we have

$$V_{d-2}(S) = \int_{C_{d-2}} G(S+t) dt < G(S) < G(S_n^d).$$

With $6 V_{d-2}(S) = d(d-1) V(S_n^d)$ we have

$$G(S_n^d) \leq \frac{d(d-1)}{6} V(S_n^d) = \frac{n+1}{6(d-2)!} 2^{2^{d-a}}.$$

The case $d=3$: Clearly: $V(S_n^3)=6(n+1)$.

To compute $G(S_n^3)$ we count the integers $x_3 \geq 0$ satisfying

$$3(n+1)x_1 + 2(n+1)x_2 + x_3 \leq 6(n+1)$$

for $(x_1, x_2) = (0, 0), (0, 1), (0, 2), (1, 0), (1, 1)$ and $(2, 0)$.

Easy computation gives $G(S_n^3) = 16n + 23$.

The case $d=4$: $V(S_1^4) = 147$.

To compute $G(S_1^4)$ we count the pairs (x_3, x_4) of integers $x_3 \geq 0, x_4 \geq 0$ satisfying

$$42x_1 + 28x_2 + 7x_3 + x_4 \leq 84$$

for $(x_1, x_2) = (0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (2, 0)$.

Counting yields $G(S_1^4) = 680$.

This completes the proof of the theorem.

We raise the following *conjecture*:

$$v(d, n) < \infty \quad \text{for all } d \geq 3 \quad \text{and } n \geq 1.$$

We remark that the conjecture implies $g(d, n) < \infty$ by Blichfeldt ([1], p.55): $G(P) \leq d! V(P) + d$ for nondegenerate lattice polytopes (compare [4]).

Similar problems may be asked for the number of i -dimensional faces of convex lattice polytopes P satisfying $G^0(P) = n$, where $0 \leq i \leq d-1$.

J. Zaks, University of Haifa and C.R.M.A., University of Montreal,
M. A. Perles, Hebrew University, Jerusalem,
J. M. Wills, University of Siegen, West Germany

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Kleine Mitteilungen

A note on the successive remainders of the exponential series

1. For real $x \neq 0$, we have by Taylor's theorem

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} \phi_n(x), \quad (1.1)$$

where

$$\phi_n(x) = e^{x\theta_n(x)}, \quad 0 < \theta_n(x) < 1. \quad (1.2)$$