An Egyptian algorithm for polynomials

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1. Introduction and notation

The Egyptian, or Ahmes, algorithm asserts that any positive proper fraction is a sum of finitely many so-called unit, or Egyptian, fractions. In other words, given positive integers $a < b$, there exist positive integers $k_1 < \cdots < k_m$ such that $a/b = 1/k_1 + \cdots + 1/k_m$. A proof that any such a/b admits such a sum representation appears in [2], p. 261, and an algorithm producing such a representation is given in [1], Exercise 4(a), p. 133. This note presents an analogous result in a kindred context, that of the polynomial ring $K[X]$, where K is a field. It will be no more difficult to consider, more generally, $R[X]$ where R is a domain with quotient field K.

2. The algorithm

It will be convenient to introduce the following terminology. A rational function $w \in K(X)$ is said to be *R-proper* in case $w = f/g$ for some nonzero $f, g \in R[X]$ such that $\deg(f) < \deg(g)$. A natural family of examples of R-proper rational functions is provided in characteristic 0 by the logarithmic derivatives g'/g arising from arbitrary $g \in K[X]$ of degree at least 1. By clearing denominators, one sees easily that w is R-proper if and only if w is K-proper. Moreover, the above definition is easily shown to be independent of the choice of f and g, by using the fact that $K[X]$ is a unique factorization domain. Such assertions, and indeed most of the calculations in this note, may be verified via the valuation properties of deg for polynomials over a domain. To wit, $\deg(g_1 g_2) = \deg(g_1) + \deg(g_2)$; and $\deg(g_1 + g_2) \leq \max(\deg(g_1), \deg(g_2))$, with equality holding if $\deg(g_1) \neq \deg(g_2)$. A particularly useful application of these properties asserts that any nonzero sum of R-proper rational functions is itself R-proper, the point being that $\deg(f_i) < \deg(g_i)$ for $i = 1,2$ implies $\deg(f_1 g_2 + g_1 f_2) < \deg(g_1 g_2)$. Observe that the analogous assertion for positive proper fractions is false: for instance, $1/2 + 1/3 + 1/4 = 13/12 > 1$.

We shall say that an *R-reciprocal representation of* a nonzero rational function $w \in K(X)$ (*of length m*) is an expression

$$w = r_1/h_1 + \cdots + r_m/h_m$$

with each $h_i \in R[X]$, $0 \neq r_i \in R$, $0 < \deg(h_1) < \deg(h_2) < \cdots < \deg(h_m)$. As any such summand r_i/h_i is R-proper, the result noted above implies that if w admits an R-reciprocal representation, then w is R-proper. (The analogue for $\mathbf{Z}$ is false: vide 13/12.) Theorem (a) will establish the converse, but first we need

Lemma. *Let f, g be nonzero elements of $R[X]$ such that* $\deg(f) < \deg(g)$. *If $r_1/h_1 + \cdots + r_m/h_m$ is an R-reciprocal representation of f/g, then* $\deg(h_1) = \deg(g) - \deg(f)$.

Proof: Multiply both sides of $f/g = \sum r_i/h_i$ by $g\prod h_i$. The result is $f\prod h_i = g \sum\limits_i r_i \prod\limits_{j\neq i} h_j$. Equating degrees, we have

$$\deg(f) + \sum \deg(h_i) = \deg(g r_1 \prod_{j\neq 1} h_j) = \deg(g) + 0 + \sum_{j\neq 1} \deg(h_j),$$

from which the conclusion is immediate.

Theorem. *Let w be a nonzero element of $K(X)$. Express w as f/g, where f and g are in $R[X]$, and $\deg(f) = n$. Then:*
(a) w is R-proper if and only if there exists an R-reciprocal representation of w.
(b) If w is R-proper, then one can algorithmically produce an R-reciprocal representation of w of length at most $n + 1$.

Proof: The 'if' half of (a) follows from the above comments. Conversely, let w be R-proper; that is, $\deg(f) < \deg(g)$. It is enough to show that w admits a K-reciprocal representation of the form $\sum 1/h_i$, of length $m \leq n + 1$. For then, by clearing denominators, $1/h_i = r_i/H_i$ for suitable $r_i \in R$ and $H_i \in R[X]$ (with $\deg(H_i) = \deg(h_i)$), and w would have an R-reciprocal representation, $\sum r_i/H_i$, of the same length m.
We shall show by strong induction on n that w has a K-reciprocal representation of the above form with length at most $n + 1$. If $n = 0$, this is clear, for f is then a constant and $w = 1/f^{-1}g$. For the induction step, use the division algorithm in $K[X]$ to write $g = h_1 f + r$, for uniquely determined h_1 and r in $K[X]$ such that $\deg(r) < n$. Note that $\deg(h_1) = \deg(g) - \deg(f) \geq 1$. Moreover,

$$\mathrm{w} = \frac{f}{g} = \frac{1}{h_1} + \frac{-r}{g h_1}. \tag{1}$$

Without loss of generality, $r \neq 0$. We claim that $v = -r/gh_1$ is K-proper. Indeed, $\deg(-r) = \deg(r) < n < \deg(g) < \deg(g) + \deg(h_1) = \deg(gh_1)$, proving the claim. Note that we have also shown

$$\deg(r) \leq \deg(g) - 2. \tag{2}$$

Now, since $\deg(r) < \deg(f)$, the induction hypothesis supplies $\sum_{i=2}^{m} 1/h_i$, a K-reciprocal representation of v, with $m - 1 \leq \deg(r) + 1 \leq n$. By (1), $w = 1/h_1 + \cdots + 1/h_m$, and so it suffices to prove that $\deg(h_1) < \deg(h_2)$. However, the lemma gives $\deg(h_2) = \deg(gh_1) - \deg(-r)$, which equals $\deg(h_1) + \deg(g) - \deg(r)$. By (2), $\deg(h_2) \geq \deg(h_1) + 2$, and the proof is complete.
As noted in [2], p. 261, the divergence of the harmonic series permits arbitrary positive rationals to be realized as sums of unit fractions. Essentially because deg is non-Archimedean, the next result is the best-possible analogue for rational functions.

Corollary. *Each element of $K(X)$ can be expressed as a sum of the form $h_0/r_0 + r_1/h_1 + \cdots + r_m/h_m$, where $r_i \in R$ for each i and $r_0 \neq 0$, $h_i \in R[X] \setminus \{0\}$ for each $i \geq 1$ and $h_0 \in R[X]$, and $\deg(h_1) < \deg(h_2) < \cdots < \deg(h_m)$.*

Proof: If $u \in K(X)$, write $u = F/g$ for suitable F, g in $K[X]$ with $g \neq 0$. By the division algorithm in $K[X]$, $F = qg + f$ for suitable q, f in $K[X]$ with $\deg(f) < \deg(g)$. Thus $u = q + f/g$. By clearing denominators, write q as h_0/r_0 for h_0, r_0 as asserted. Finally, express f/g as $r_1/h_1 + \cdots + r_m/h_m$, either with an R-reciprocal representation of f/g (if $f \neq 0$) or trivially with vanishing numerators (if $f = 0$).

Remark: (a) If $R = K$ then each R-proper rational function has a 'unit' R-reciprocal representation, that is, one in which each numerator $r_i = 1$. The converse also holds. Indeed, if s is a nonzero element of R and s/X admits an R-reciprocal representation of the type $\sum h_i^{-1}$, clearing denominators leads to

$$s \prod_{i=1}^{m} h_i = X \sum_i \Big(\prod_{j \neq i} h_j\Big).$$

Then, if a_i denotes the leading coefficient of h_i, equating the leading coefficients of the left- and right-sides of the display yields $s a_1 a_2 \cdots a_m = a_2 \cdots a_m$. By cancellation, $s a_1 = 1$; that is, $s^{-1} \in R$.

(b) It seems natural to consider power series analogues of the above theorem. Here, we record only the simple fact that $K[[X]]$ admits such an analogue with $m = 1$. More generally, let V be the valuation ring (domain) of a (discrete rank one) valuation v with value group $\mathbf{Z}$ and local uniformizing parameter π. Then any 'proper' element, f/g with f and g in V and $v(f) < v(g)$, can be expressed as $1/\pi^{v(g)-v(f)} u$ for an appropriate unit u of V.

(c) Unit reciprocal representations need not be unique. To see this in classical context of the Egyptian algorithm, observe that the proper fraction $7/12$ is both $1/3 + 1/4$ and $1/2 + 1/12$. An analogue for rational functions (for simplicity, with $R = \mathbf{Q}$) is easily illustrated:

$$(X^2 + X + 1)/X^3 = 1/X + 1/X^2 + 1/X^3 = 1/(X-1) + 1/X^3(-X+1).$$

It is a bit harder to illustrate nonuniqueness with distinct $\mathbf{Q}$-reciprocal representations sharing a common initial term and having the same length. For instance, one has $\mathbf{Q}$-reciprocal representations

$$\begin{aligned} \frac{X^5 + X^4 + X^3 + X^2 + X - 1}{X^6 + X^3 - X^2} &= \frac{1}{X} + \frac{1}{X^2} + \frac{1}{X^3 - X} + \frac{1}{p_6} + \frac{1}{p_{12}} \\ &= \frac{1}{X} + \frac{1}{X^2 - X} + \frac{1}{-X^4 + 2X^2} + \frac{1}{p_8} + \frac{1}{p_{16}} \end{aligned} \tag{3}$$

where p_t denotes a suitable polynomial of degree t.

To indicate the genesis of (3), we first reveal a general way to synthesize a rational function admitting distinct K-reciprocal representations having a common initial term. Consider two distinct polynomials, f and h_1 of the same degree $n \geq 2$ and with the same term of highest degree; let r be a polynomial of degree at most $n-1$ and set $g = h_1 f + r$. We know from the proof of the theorem that f/g has a K-reciprocal representation with initial term $1/h_1$. Does it also have one whose initial term is $1/f$? By the lemma and theorem, it will since $\deg(f) + \deg(g) - \deg(f^2 - g) > n$, the point being that f^2 and g have the same leading term. One thereby obtains distinct K-reciprocal representations, $1/h_1 + \cdots = 1/f + \cdots$; adding $1/X$ to (the left of) each produces distinct such representations with the same initial term.

The two representations given by the above recipe may, however, have unequal lengths. Consider, for instance, the choices $f = X^2 + 1$, $h_1 = X^2 - 1$, $r = X$. The recipe produces

$$\begin{aligned}\frac{X^4 + X^3 + 2X - 1}{X^5 + X^2 - X} &= \frac{1}{X} + \frac{1}{X^2 - 1} + \frac{1}{-X^5 + X^3 - X^2 + X + 1} + \frac{1}{p_{11}} \\ &= \frac{1}{X} + \frac{1}{X^2 + 1} + \text{sum of three additional terms.}\end{aligned} \tag{4}$$

Let w denote the rational function in (4), we compute via the first equation in (4) that

$$\begin{aligned}v = 1/X + w/X &= \frac{X^5 + X^4 + X^3 + X^2 + X - 1}{X^6 + X^3 - X^2} \\ &= \frac{1}{X} + \frac{1}{X^2} + \frac{1}{X^3 - X} + \frac{1}{-X^6 + X^4 - X^3 + X^2 + X} + \frac{1}{p_{12}},\end{aligned}$$

which explains the origin of the first equation in (3). How does one obtain the second equation in (3)? Simply apply the Theorem's algorithm to w, multiply the result through termwise by $1/X$, and then add $1/X$ to (the left of) the ensuing expression. Finally, it is amusing to note that an application of the Theorem's algorithm directly to v produces yet another distinct $\mathbf{Q}$-reciprocal representation of v.

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Berichtigung

Correction to paper: Packing of 180 equal circles on a sphere. Elemente der Mathematik. Vol. 38, 1983, 119–122

Professor H. S. M. Coxeter kindly drew my attention to the fact that figures of packings of 72 and 180 circles are chiral and not centro-symmetric. Namely, central symmetry can occur in tessellation $\{3, q+\}_{b,c}$ ($q = 4$ or 5) if $bc(b - c) = 0$, that is, the tesselation has a plane of symmetry. Thus, the statement in the last sentence of the paper is not valid.

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