Construction of a line through a given point to divide a triangle into two parts with areas in a given ratio

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1. Introduction

A ruler and compass construction for a line through a given point dividing a triangle into two parts with areas in a given ratio seems not to be readily available although it must surely be well known: it is mentioned, for example, on pages 76–77 of a report on the Teaching of Geometry [1] prepared for the Mathematical Association. The construction set out in this paper serves as a prelude to a discussion of whether it is possible to construct the required line through a particular point and, when it is possible to do so, of the number of such lines.

Suppose the problem is to cut off triangle $C\bar{P}\bar{Q}$ from triangle ABC by a line $O\bar{P}\bar{Q}$ through O in such a way that the area of $C\bar{P}\bar{Q}$ is n times that of ABC. In figure 1, let D, E divide BC, CA respectively so that $CD : CB = CE : CA = n : 1$. Then the set of points $\{P\}$ of the interval AE can be put into one-to-one correspondence with the set of points $\{Q\}$ of the interval DB in such a way that the line PQ in each case divides the triangle so that $\Delta CPQ : \Delta ABC = n : 1$. The pencils of rays $O\{P\}, O\{Q\}$ are similarly in one-to-one correspondence and the problem is to find the particular rays $O\bar{P}, O\bar{Q}$ which coincide and so give the line $O\bar{P}\bar{Q}$ which has both the required properties of passing through O and of dividing triangle ABC in the required ratio.

In fact there is a projectivity between the pencils $O\{P\}, O\{Q\}$ so the standard method of finding the self-corresponding members of these two pencils can be used to identify the required ray $O\bar{P}\bar{Q}$. This method is described, for example, in [3] where, in a set of problems following the construction of the double points of two projective ranges on the same line, there appears a problem equivalent to the problem under discussion, namely the construction of a line through a given point to include with two given lines a triangle of given area.

It is now possible to examine the question of how many of the required transversals can be drawn through given points in the plane and indeed whether any such transversals can be drawn. In the case $n = 1/2$ when the triangle is bisected it turns out that it is always possible to draw at least one transversal through a given point; from points on three arcs of hyperbolas joining the mid-points of the medians of the triangle and tangential to those medians, it is possible to draw two such transversals; and from all points inside the 'triangle' formed by these arcs, three transversals can be drawn. Similar but more complex results are obtained when the triangle is divided into two parts of unequal area in the ratio $n : (1 - n)$ and it becomes necessary to allow for the triangular portion to have area of either n times or $(1 - n)$ times that of triangle ABC. In particular, it will be shown that for certain values of n there are points through which no transversal can be drawn while for other values of n there are points through which as many as six transversals can be drawn.

This part of the investigation overlaps with results described by Derek Ball [4]. He shows, for example, that what he calls the 'halving envelope' of an equilateral triangle i.e. the envelope of lines in the plane which bisect the triangle is 'a concave "triangle" made up of the arcs of three hyperbolae; the vertices of this triangle lie at the mid-points of the medians. This is the locus of points from which two bisectors can be drawn and separates the inner region from which three bisectors can be drawn from the outer region from which one bisector can be drawn through each point. Ball also gives the envelope of the transversals which divide the triangle in an arbitrary ratio as a combination of six hyperbolic arcs which form various patterns as the ratio varies. These arcs enter the discussion of the general problem in the same way as the hyperbolic 'triangle' enters the bisection problem.

2. Construction based on finding self-corresponding rays of two projective pencils

By way of example, consider the problem of finding a line through a given point O which cuts the sides CA, CB in $\bar{P}, \bar{Q}$ respectively so that $\Delta C\bar{P}\bar{Q} : \Delta ABC = n : 1$. If D, E on CB, CA respectively are chosen so that $CD : CB = CE : CA = n : 1$, it is clear that AD and BE are two lines which divide the triangle in the required way, each of the triangles CAD, CBE having area equal to n times the area of triangle ABC (fig. 2). Moreover if P is any point of the interval AE and Q is the point of the interval BD such that AQ is parallel to DP, then PQ also is a line which divides the triangle in the required way: for $\Delta CPQ = \Delta CPD + \Delta DPQ = \Delta CPD + \Delta DPA = \Delta CAD = n\Delta ABC$. (It may be noted that Q can equally well be constructed by drawing EQ parallel to BP.)

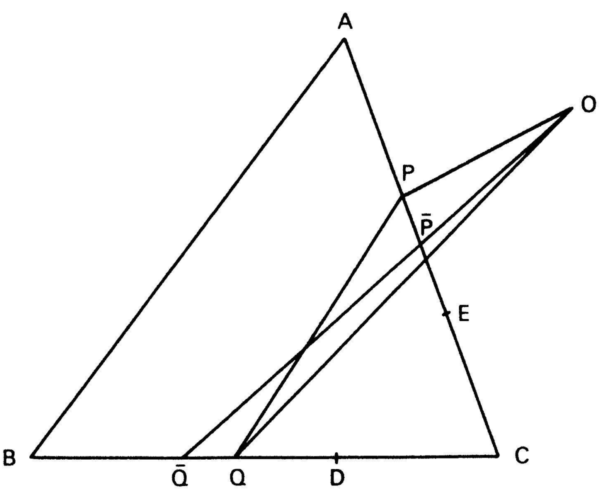


Figure 1

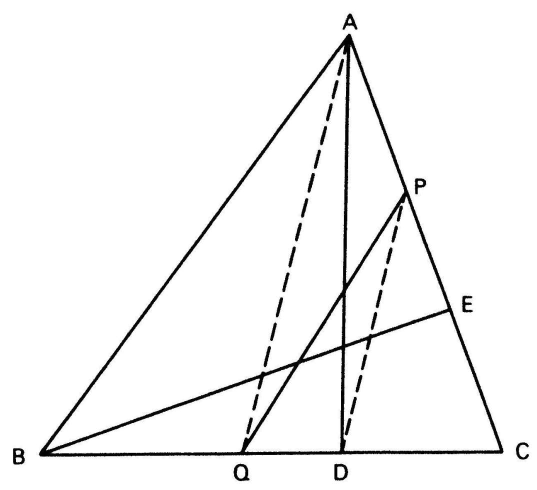


Figure 2

Thus the points $\{P\}$ of AE are in one-to-one correspondence with the points $\{Q\}$ of DB, with A corresponding to D and E to B, in such a way that the line PQ divides the triangle so that $\Delta CPQ = n\Delta ABC$ and all such lines are members of the set $\{PQ\}$. Joining the points of these ranges to the fixed point O gives two pencils of lines $O\{P\}, O\{Q\}$ which are in one-to-one correspondence and it can be seen that the problem will be solved if these two pencils have a self-corresponding member i.e. if there is a point $\bar{P}$ on AE which corresponds to $\bar{Q}$ on DB in such a way that $O\bar{P}$ and $O\bar{Q}$ coincide (fig. 1). In this case there will be a line $\bar{P}\bar{Q}$ which passes through O and cuts off triangle $C\bar{P}\bar{Q}$ of area equal to n times that of triangle ABC.

The question now arises of whether it is possible, for a given point O, to find such a self-corresponding member of the pencils $O\{P\}, O\{Q\}$ i.e. whether there is a member of the set $\{PQ\}$ which passes through O. It is clear that at least one of the lines PQ will pass through any point O which lies in that portion of the plane which is bounded by the (infinite) lines AD and BE and includes the intervals AE and DB. It will be shown later that in fact there is a unique line PQ through every such point and that there are other points in the plane through which one and in some cases two of the lines PQ will pass. Consideration of the six cases in which the transversal cuts off from one of the three vertices of triangle ABC a triangle of area either n or $(1-n)$ times that of triangle ABC leads to a determination of the number of transversals which can be drawn from any given point to divide the triangle into two parts with areas in the ratio $n:(1-n)$.

First, however, there is the question of finding the self-corresponding line in the pencils $O\{P\}, O\{Q\}$ on the assumption that O is a point for which such a line exists. For this purpose, it is convenient to consider the problem in the projective plane, introducing a line at infinity, l_∞. In this plane (fig. 3), the lines AB and DE, parallel in the Euclidean plane, intersect at the point C' on l_∞. The construction of the point Q on DB corresponding to P on AE involves joining DP to cut l_∞ in X and then joining XA to cut BC in Q (alternatively, BP can be joined to cut l_∞ in Y and YE to cut BD in Q). It is now apparent that the two ranges $\{P\}$ on AC and $\{Q\}$ on BC, hitherto described simply as in one-to-one correspondence, are in fact projective ranges, being linked by the following series of perspectivities:

$$\{Q\} \overset{A}{\underset{\wedge}{=}} \{X\} \overset{D}{\underset{\wedge}{=}} \{P\}.$$

The problem is therefore to find the self-corresponding members of the two projective pencils $O\{P\}, O\{Q\}$, assuming such self-corresponding members do exist. There are at most two such self-corresponding members and if it is to be possible to draw the required transversal through the given point O, at least one of these must intersect CA and CB in points $\bar{P}$ and $\bar{Q}$ which lie in the segments AE and BD respectively.

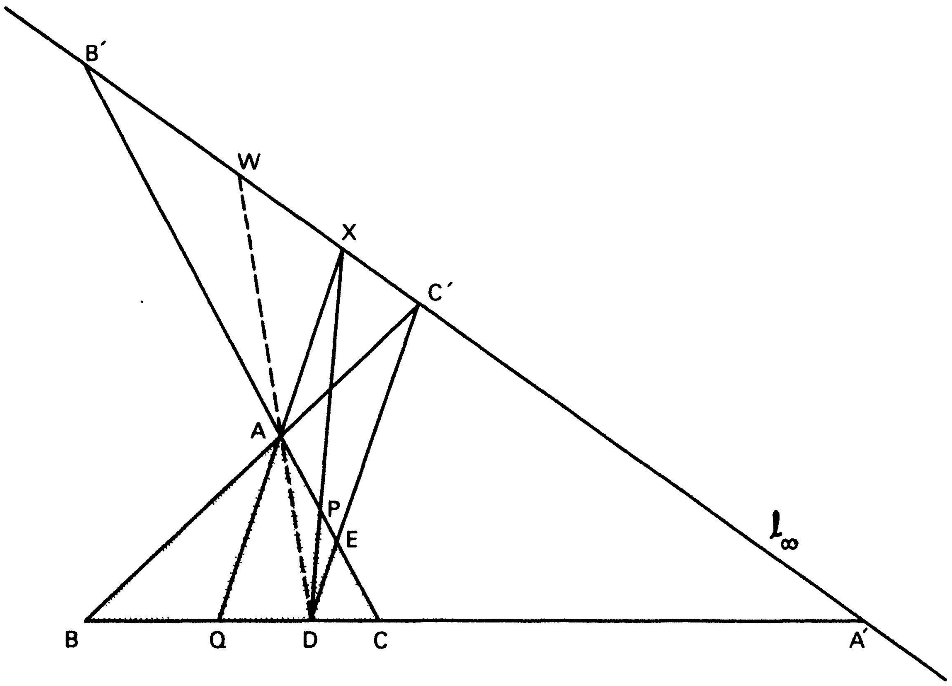


Figure 3

There is a standard construction for finding these self-corresponding lines (see, for example [3], [5]) which requires only that the projectivity be defined by the correspondence between three pairs of lines which are determined by the fixed points of the problem, namely A, B, C, D, E, O, and l_∞. These three pairs of lines are immediately to hand for (with points on l_∞ as defined in figure 3)

$$O\,(QBDCA') \overset{A}{\underset{\wedge}{=}} O\,(XC'WB'A') \overset{D}{\underset{\wedge}{=}} O\,(PEAB'C).$$

Thus for example, $O\,(B,E)$, $O\,(D,A)$, $O\,(C,B')$ are three pairs of corresponding lines, all defined by the fixed points of the problem.

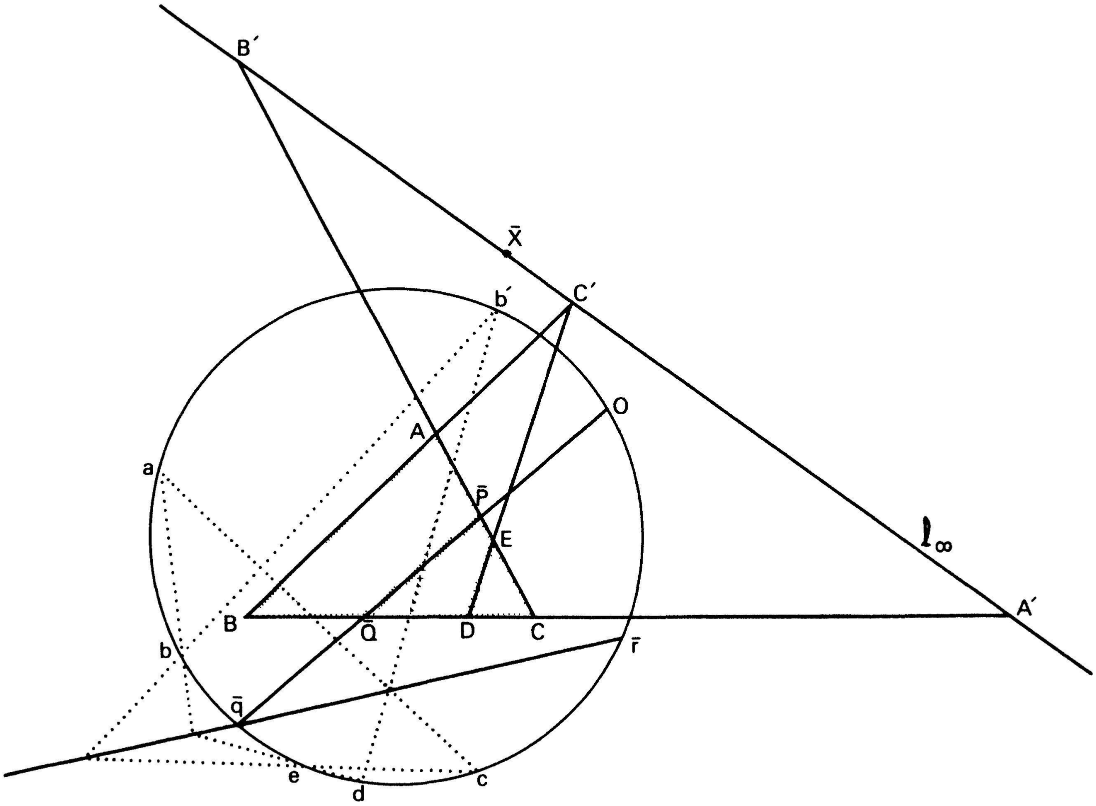


Figure 4

To construct the self-corresponding rays of the projective pencils $O\{P\}, O\{Q\}$, take an arbitrary conic (such as a circle) through the point O which cuts the lines $O\,(B,D,C,E,A,B')$ in the points b,d,c,e,a,b' respectively (fig. 4). The Pascal line of the hexagon $bdcb'ae$ i.e. the line through the three points $(ab,de), (ac,db'), (ec,bb')$ will cut the conic in two points, touch it at one point or not cut it at all. The self-corresponding lines of the projectivity between $O\{P\}$ and $O\{Q\}$, when they exist, are obtained by joining O to the points of intersection of the Pascal line and the conic. If, as in figure 4, there is one of these points $\bar{q}$ such that the self-corresponding line $O\bar{q}$ which it determines cuts BC in a point $\bar{Q}$ of DB and AC in a point $\bar{P}$ of AE, then this line will be a transversal which divides the triangle in the required way. It will follow (and provide a check for the construction) that if $D\bar{P}$ cuts l_∞ in $\bar{X}$, then $\bar{X}, A, \bar{Q}$ are collinear.

It remains now to translate the construction back into the Euclidean plane. To construct the required line through O, draw an arbitrary circle through O. Let $O\,(B, D, C, E, A)$ cut this circle in the points b, d, c, e, a respectively and let the line through O parallel to AC cut this circle in b'. Proceed exactly as before to construct the self-corresponding line or lines or to find that there are none. If, as in figure 5, a self-corresponding line cuts the sides BC, AC in $\bar{Q}$ and $\bar{P}$, points in the segments DB and AE respectively, then $O\bar{P}\bar{Q}$ is a transversal which divides the triangle into two parts with areas in the required ratio. In this case a check on the construction is provided by the requirement that $D\bar{P}$ and $A\bar{Q}$ be parallel.

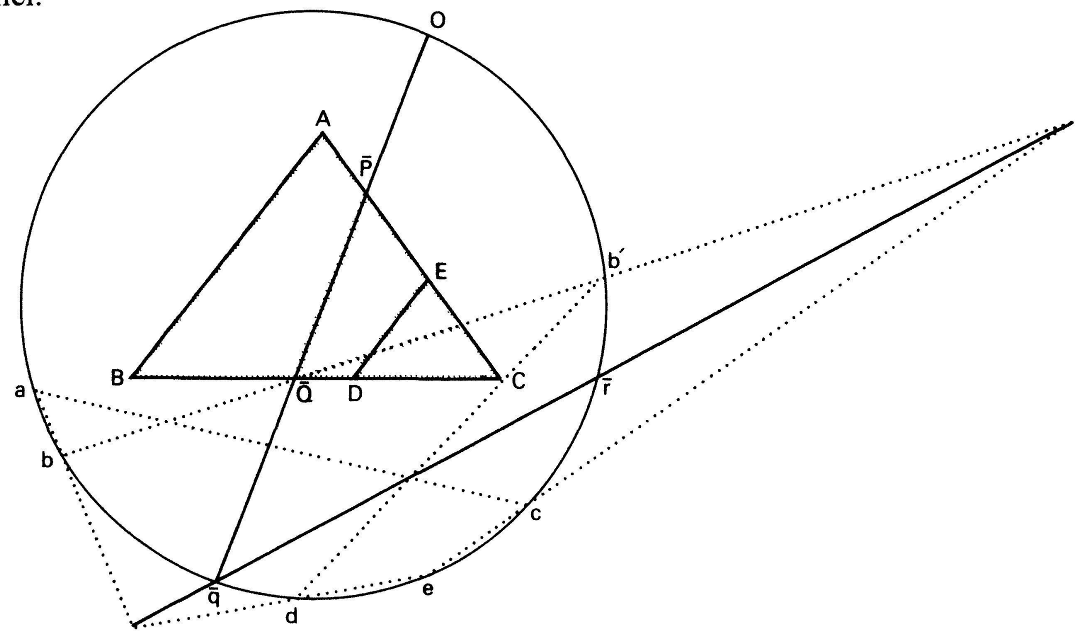
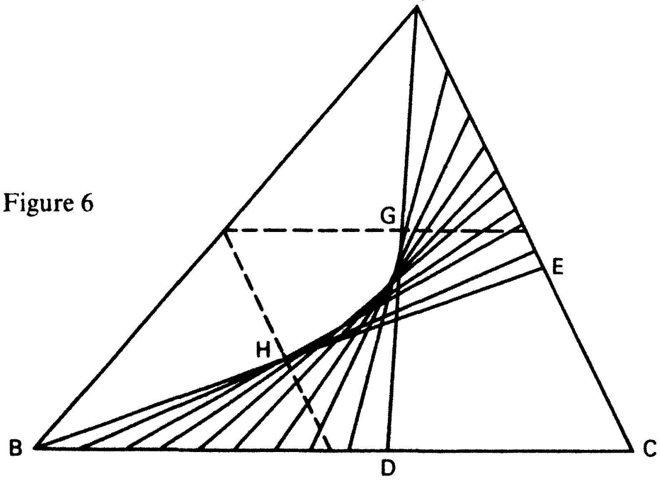


Figure 5

3. For what points O can the construction be carried out? How many transversals will pass through a given point O?

It has been noted already that the construction just described will produce a transversal $O\bar{P}\bar{Q}$ with the required properties whenever O lies in the region bounded by the lines AD, BE which includes the intervals AE, BD. This follows from the fact that at least one of the lines PQ which cuts off triangle CPQ of area n times that of triangle ABC will pass through any point in this region.

Figure 6



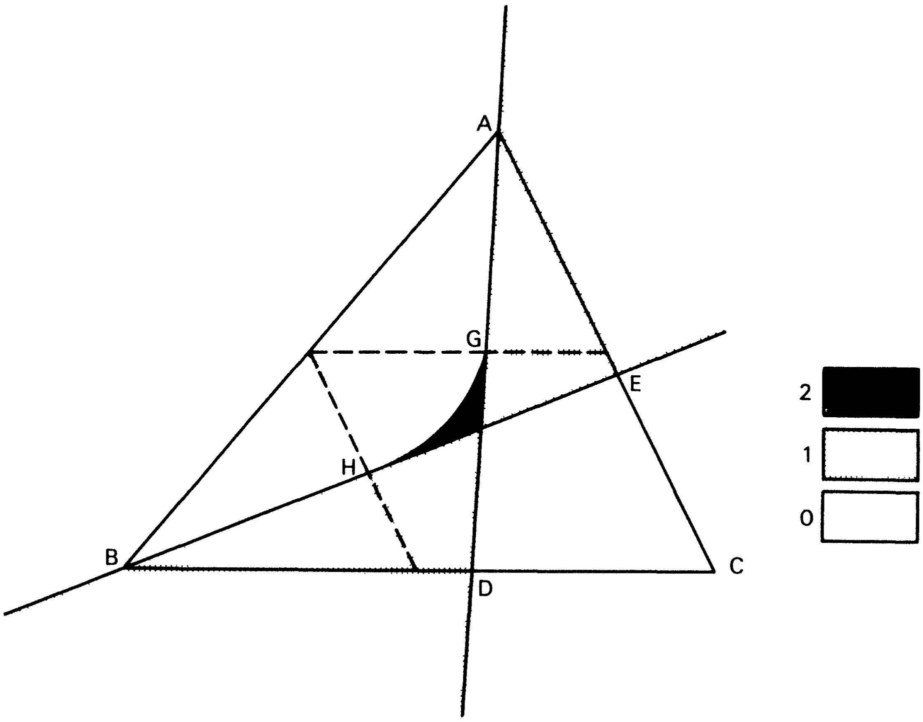


Figure 7

It will be shown in the next section that these lines PQ envelop the arc of a hyperbola with its centre at C, CA and CB as asymptotes and its points of contact with the two tangents DA, EB at the mid-points G, H of those intervals (fig. 6). Assuming this result, it can be seen that discussing whether one or more of the transversals PQ pass through a given point O is equivalent to considering whether one or more tangents to this hyperbolic arc pass through O. This leads to the conclusion that through all points O previously considered, namely all points in the region bounded by AD and BE and including the intervals AE and BD, and through all points of the hyperbolic arc GH, there is just one such line; and through interior points of the 'triangle' bounded by the arc and its tangents at G and H (i. e. AD and BE) and the points of its straight line boundaries, there are two such lines.

This pattern for transversals through O which cut off triangle $C\bar{P}\bar{Q}$ of area equal to n times the area of triangle ABC is shown in figure 7 in which the shading indicates the number of transversals which can be drawn from a point in each region of the plane. To obtain the full story for transversals which divide the triangle in the ratio $n : (1 - n)$, it is convenient to restrict n so that $0 < n \leq 1/2$ and to consider two cases for each vertex in which the triangular portion has area n or $(1 - n)$ times that of triangle ABC, except when $n = 1/2$ and the two cases are identical. The patterns corresponding to these six cases (three when $n = 1/2$) are then superposed to find how many transversals can be drawn from any point in the plane and how to divide the plane into regions according to the number of transversals which can be drawn from each point of the region.

The simplest case is the problem in which the triangle is bisected and $n = 1/2$. If the mid-points of the sides of triangle ABC are named D, E, F and the mid-points of the sides of triangle DEF are named G, H, J (fig. 8), the regions which must be superposed are defined by three pairs of lines and their corresponding hyperbolic arcs: BE, CF, HJ;

CF, AD, JG; AD, BE, GH. Considering the numbers of transversals through each point of the plane for each of these three cases leads to the result indicated in figure 8 in which three transversals can be drawn through each interior point of the 'triangle' GHJ and one transversal through each exterior point. (From points on the boundary, two transversals can be drawn.)

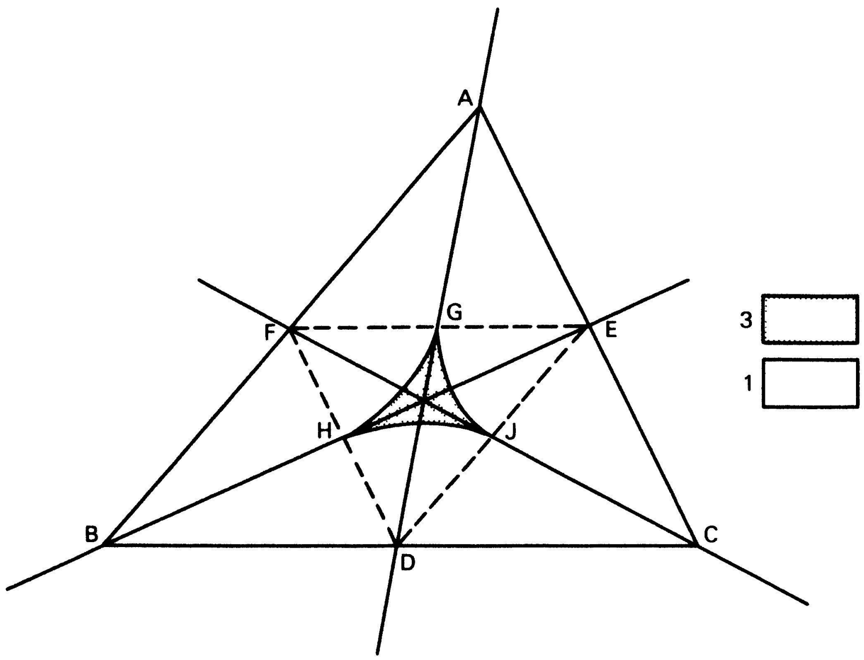


Figure 8

When n lies between 1/2 and 0, there are two transversals through each vertex of the triangle e.g. AD_1, AD_2 which intersect EF in G_1, G_2 respectively; the corresponding notation for the other vertices is shown in figure 9 (also 10, 11). In these cases there are six pairs of lines and associated hyperbolic arcs which can be sufficiently identified by naming the arcs: H_1J_2, H_2J_1; J_1G_2, J_2G_1; G_1H_2, G_2H_1. Derek Ball [4] has discussed how these

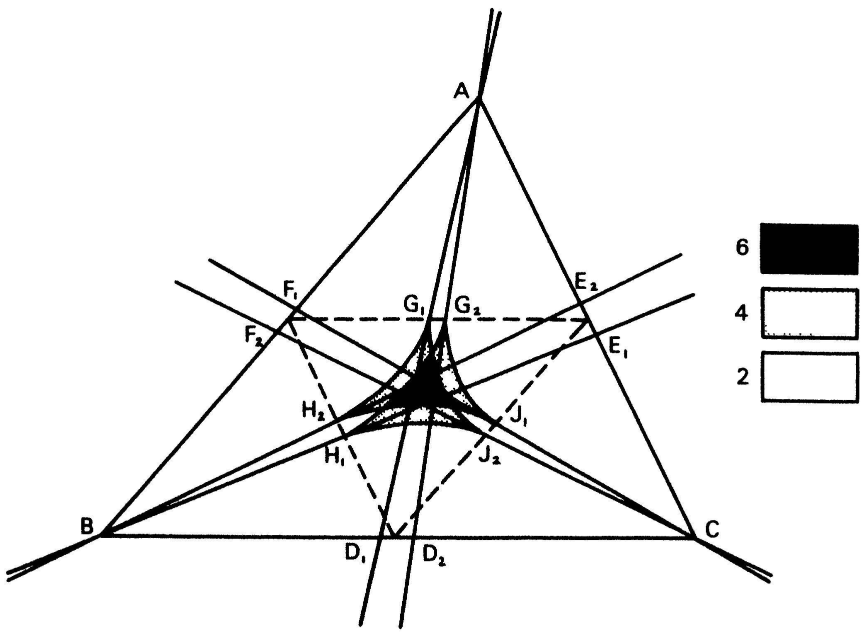


Figure 9

arcs combine to outline interestingly shaped regions of the plane. As n decreases from $1/2$, there appears first the shape indicated in figure 9 which Ball describes as a double triangle: there is a central 'triangular' region joined across its sides to three 'quadrilaterals'. Then, as n decreases further, the central 'triangle' reduces to a point and the arcs form three 'triangles' as in figure 10. Finally, as n takes still smaller values, a central 'triangle' opens up again and is now joined at its vertices to three other 'triangles' as in figure 11.

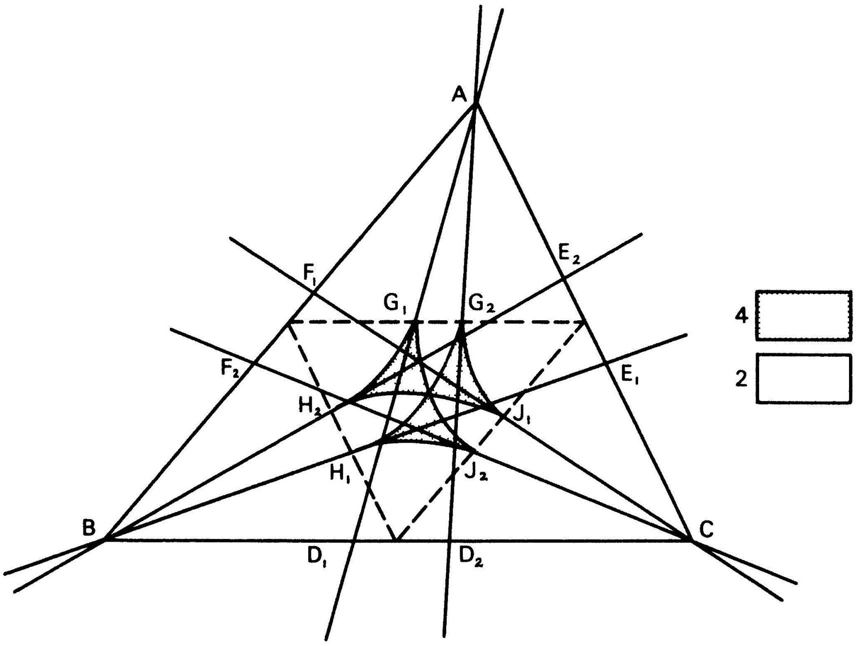


Figure 10

When a census is taken of the number of transversals through each point of the plane arising from each of the six regions associated with a pair of lines such as BE_1, CF_2 and the arc H_1J_2, the results are as indicated in figures 9–11. These show a progression from figure 9, which can perhaps be thought of as produced by a double exposure of figure 8 with a slight move of the subject between shots, through figure 10 to figure 11. It will be observed that from points of the central region which appears in figure 9 we can draw six transversals while from the central region which re-appears in figure 11 no transversals at all can be drawn. In all cases, the number of transversals which can be drawn from the points of a boundary between two regions is the arithmetic mean of the numbers which can be drawn from the regions.

4. The envelope of the lines $\{PQ\}$

It remains to show that the envelope of the lines $\{PQ\}$ is indeed a hyperbola with properties assumed in section 3.

Since the lines $\{PQ\}$ are the joins of the points of the projective ranges $\{P\}$ on AC and $\{Q\}$ on BC, their envelope is certainly a conic which has AC and BC themselves as tangents. The points of contact of these lines with the conic are those points of $\{Q\}$ and $\{P\}$ respectively which correspond to the point C. It has already been noted that

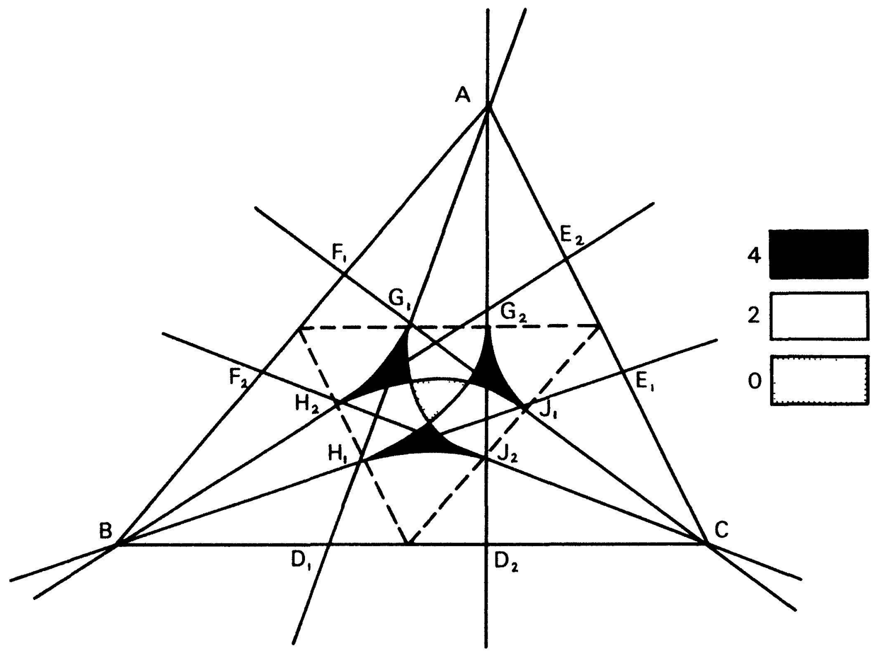


Figure 11

$$(PEAB'C) \,\bar{\wedge}\, (QBDCA')$$

so these points are the points A' and B' of the line at infinity. It follows that the conic is a hyperbola with CA and CB as asymptotes.

The part of the hyperbola of interest lies between the points of contact of BE and AD so these points must be found. For example, the point of contact of BE with the hyperbola is found by obtaining the range of points on BE which is projective with each of the ranges $\{Q\}$ on BC and $\{P\}$ on AC and then identifying the point of that range which corresponds to B as a point of $\{Q\}$.

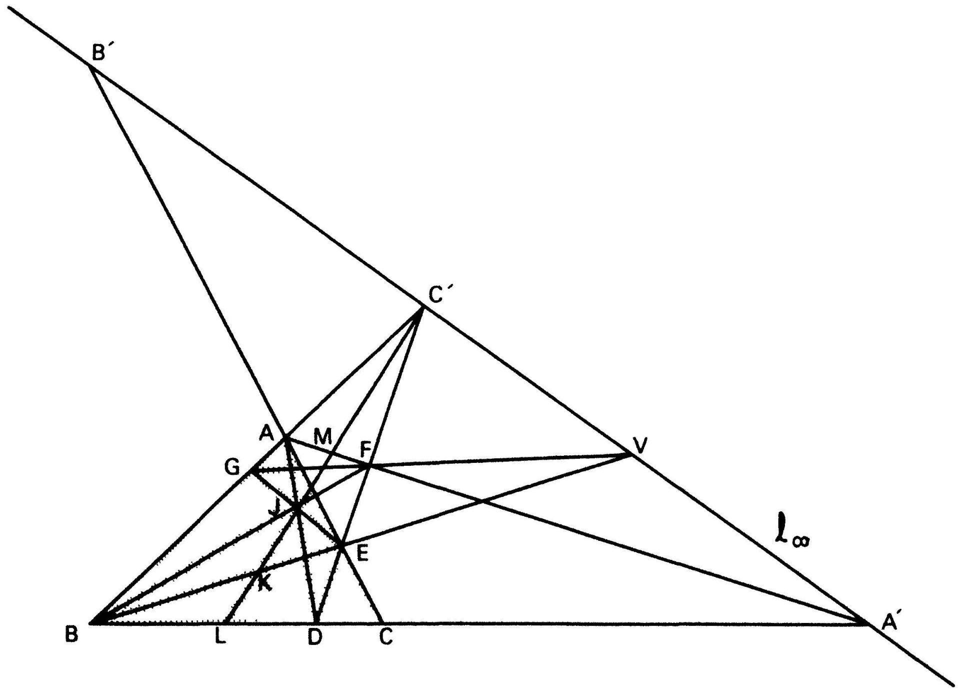


Figure 12

To this end, continue as in figure 12 the construction started in figure 3: draw BE to cut l_∞ in V, AA' to cut $C'D$ in F, VF to cut AB in G. If AD and BF meet in J and $C'J$ cuts BE, BC, AF in K, L, M respectively, it is clear from the quadrangle $AFDB$ that $(BLDA')$ is a harmonic range and accordingly that the pencil C' $(BLDA')$ is harmonic. Consideration of the quadrangle $BGFE$ then shows that GE and BF intersect on $C'L$ and hence meet in J. It can now been seen that

$$EAB'C \overset{D}{\doublebarwedge} C'A?B \overset{J}{\doublebarwedge} K??B$$

(where ? indicates a point which is not labelled). Since it is known that $(BDCA') \barwedge (EAB'C)$, it follows that $(BDCA') \barwedge (K??B)$ so that K on BE corresponds in the projectivity to B on BC (and B on BE corresponds to A' on BC). This shows that K is the point of contact with the hyperbola of the tangent BE (and confirms that BC touches the hyperbola at C').
Again, because

$$BDCA' \overset{E}{\doublebarwedge} BC'A? \overset{M}{\doublebarwedge} ?JA?$$

and $(EAB'C) \barwedge (BDCA')$, it follows that $(EAB'C) \barwedge (?JA?)$ so that A as a point of AC corresponds to J as a point of AD (and B' as a point of AC corresponds to A as a point of AD). Hence the point of contact with the hyperbola of the tangent AD is the point J (and B' is the point of contact of AB).
Considering the intersections of AD and BE respectively with the harmonic pencil C' $(BLDA')$ shows that the harmonic conjugate of J with respect to A, D is a point on l_∞ and of K with respect to B, E is the point V on l_∞. In the Euclidean plane therefore, the points of contact with the hyperbola of AD and BE are the mid-points of these intervals.

ACKNOWLEDGMENTS

I am grateful to a number of colleagues for useful discussions, particularly L. G. Kovács with whom I have worked on the problem [2] which caused me to become interested in the problem of this paper and J. C. Barton who provided reference [1].
When I sought advice on a preliminary version of the paper from Professor H. S. M. Coxeter, he referred me to Professor J. G. Mauldon who produced a 1938 manuscript with a solution by essentially the same method that I had proposed. I have had most valuable correspondence and discussion with Professor Mauldon and I gratefully acknowledge that the construction given in section 2 of this paper owes much to his 1938 presentation.

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