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LITERATURVERZEICHNIS

1 Blaschke W.: *Kinematik und Quaternionen* (VEB Deutscher Verlag der Wissenschaften, Berlin 1960).
2 Krames J.: *Über Fusspunktkurven von Regelflächen und eine besondere Klasse von Raumbewegungen (Über symmetrische Schrotungen I).* Monatsh. Math. Phys. *45*, 394–406 (1937).
3 Tarnai T. and Makai E. jr.: *A Movable Pair of Tetrahedra.* Im Druck.

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A new geometric inequality

Let $\omega \in (0, \pi)$ be defined by the equation

$$\cot \omega = \cot \alpha_1 + \cot \alpha_2 + \cot \alpha_3 \tag{1}$$

where $\alpha_1, \alpha_2, \alpha_3$, are positive numbers satisfying

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi . \tag{2}$$

If α_1, α_2 and α_3 are interpreted as the three angles of a triangle (T), then ω is called the Brocard angle of (T) and there exists a number of identities relating ω and α_1, α_2 and α_3 [4]. This note is concerned with the problem of finding inequalities between ω and α_1, α_2 and α_3. Since the appearance of [1], this problem has received much attention. At present the following inequalities are known [1–3].

$$2\omega \leqq \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3) = \frac{\pi}{3} . \tag{3}$$

This is the oldest known inequality and follows from the inequality $\cot^2 \omega \geqq 3$ which is readily obtained from (1). The next inequality is

$$2\omega \leqq \sqrt[3]{\alpha_1 \alpha_2 \alpha_3} \tag{4}$$

which was proved in [1]. It is sharper than (3).
In [2] it was shown that

$$\omega^3 \leqq (\alpha_1 - \omega)(\alpha_2 - \omega)(\alpha_3 - \omega) , \tag{5}$$

an inequality that implies (4).
Using the method of Lagrange multipliers, Mascioni [5] proved the inequality

$$2\omega \leqq 3\left(\sum 1/\alpha_i\right)^{-1} . \tag{6}$$

This inequality is sharper than (4), since the harmonic mean of three numbers is less than or equal to their geometric mean. A different proof of (6) appears in [3]. Since

(5) implies that ω is less than or equal to the geometric mean of $\alpha_1-\omega$, $\alpha_2-\omega$ and $\alpha_3-\omega$, it is natural to ask how ω is related to their harmonic mean. In the present note we prove

Theorem 1. *If ω is defined by (1) and (2) then*

$$\omega \geqq 3\{\sum 1/(\alpha_i-\omega)\}^{-1}. \tag{7}$$

Equality holds if and only if $\alpha_1=\alpha_2=\alpha_3=\pi/3$.

Proof: Let $f(x)=\cot x-\dfrac{1}{x}$ where $0<x<\pi$.

Then $f'(x)=-\dfrac{1}{\sin^2 x}+\dfrac{1}{x^2}$ and $f''(x)=2\csc^3 x\left\{\cos x-\left(\dfrac{\sin x}{x}\right)^3\right\}$. Since $\sin x<x$ for $0<x<\pi$ we have $f'(x)<0$. In [5], it was shown that $\left(\dfrac{\sin x}{x}\right)^3>\cos x$ in $(0,\pi)$. Then $f''(x)<0$ in $(0,\pi)$. It follows that f is a concave decreasing function in $(0,\pi)$. In particular, if $x_1, x_2, \ldots, x_6$ are six numbers in $(0,\pi)$, then

$$f\left(\frac{x_1+x_2+\ldots+x_6}{6}\right)\geqq\frac{1}{6}\{f(x_1)+f(x_2)+\ldots+f(x_6)\}.$$

If this inequality is used with the six numbers $\alpha_1-\omega$, $\alpha_2-\omega$, $\alpha_3-\omega$, ω, ω, ω, all of which lie in $(0,\pi)$ we obtain

$$\sum_{i=1}^{3}\left\{\cot(\alpha_i-\omega)-\frac{1}{\alpha_i-\omega}+\cot\omega-\frac{1}{\omega}\right\}\leqq 6\{\cot(\pi/6)-(6/\pi)\}$$

$$\leqq 6\left\{\cot\omega-\frac{1}{\omega}\right\}, \tag{8}$$

where the last inequality follows because $\omega\leqq\pi/6$ and $\cot x-1/x$ is decreasing in $(0,\pi)$.

From (8) we obtain

$$\sum_{i=1}^{3}\{\cot(\alpha_i-\omega)-\cot\omega\}+3/\omega\leqq\sum_{i=1}^{3}\frac{1}{\alpha_i-\omega}. \tag{9}$$

The proof of (7) will be complete if we show that the sum on the left-hand side of (9) is positive. This is the difficult part of the proof. It depends, in part, on the following identities satisfied by ω. Their derivation is quite easy.

(i) $\cot\alpha_1\cot\alpha_2+\cot\alpha_2\cot\alpha_3+\cot\alpha_3\cot\alpha_1=1$;

(ii) $\prod_{i=1}^{3}(\cot\omega-\cot\alpha_i)=\prod_{i=1}^{3}\csc\alpha_i$;

(iii) $\sum_{i=1}^{3} \csc^2 \alpha_i = \csc^2 \omega$; (10)

(iv) $\sum_{i=1}^{3} \csc^4 \alpha_i + 4 \cot \omega \prod_{i=1}^{3} \csc \alpha_i = \csc^4 \omega$.

We now consider the sum on the left-hand side of (9). We have

$$\cot(\alpha_i - \omega) - \cot \omega = \frac{\cot \alpha_i \cot \omega + 1}{\cot \omega - \cot \alpha_i} - \cot \omega = \frac{-\cot^2 \omega + 2 \cot \omega \cot \alpha_i + 1}{\cot \omega - \cot \alpha_i}$$

$$= -(\cot \omega - \cot \alpha_i) + \frac{\csc^2 \alpha_i}{\cot \omega - \cot \alpha_i}.$$

Thus

$$\sum_{i=1}^{3} \{\cot(\alpha_i - \omega) - \cot \omega\} = -2 \cot \omega + \sum_{i=1}^{3} \frac{\csc^2 \alpha_i}{\cot \omega - \cot \alpha_i}. \tag{11}$$

From (10; (i)) we have

$$\cot \alpha_1 \cot \alpha_2 = 1 - \cot \alpha_3 (\cot \alpha_1 + \cot \alpha_2) = 1 - \cot \alpha_3 (\cot \omega - \cot \alpha_3)$$

$$= \csc^2 \alpha_3 - \cot \alpha_3 \cot \omega.$$

Thus

$$(\cot \omega - \cot \alpha_1)(\cot \omega - \cot \alpha_2)$$

$$= \cot^2 \omega - \cot \omega (\cot \alpha_1 + \cot \alpha_2) + \csc^2 \alpha_3 - \cot \alpha_3 \cot \omega = \csc^2 \alpha_3. \tag{12}$$

A similar formula holds for $\csc^2 \alpha_1$ and $\csc^2 \alpha_2$. Returning to the second sum in (11) and using ((10); (ii)) and (12) we obtain

$$\sum_{i=1}^{3} \frac{\csc^2 \alpha_i}{\cot \omega - \cot \alpha_i} = \frac{1}{\prod_{i=1}^{3} (\cot \omega - \cot \alpha_i)} \sum_{i=1}^{3} \csc^4 \alpha_i = \frac{1}{\prod_{i=1}^{3} \csc \alpha_i} \sum_{i=1}^{3} \csc^4 \alpha_i. \tag{13}$$

If we use (13) in (11) and then use ((10); (iv)) we obtain

$$\sum_{i=1}^{3} \{\cot(\alpha_i - \omega) - \cot \omega\} = \frac{1}{\prod_{i=1}^{3} \csc \alpha_i} \left\{ \sum_{i=1}^{3} \csc^4 \alpha_i - 2 \cot \omega \prod_{i=1}^{3} \csc \alpha_i \right\}$$

$$= \frac{1}{\prod_{i=1}^{3} \csc \alpha_i} \left\{ \sum_{i=1}^{3} \csc^4 \alpha_i + \frac{1}{2} \sum_{i=1}^{3} \csc^4 \alpha_i - \frac{1}{2} \csc^4 \omega \right\} = \frac{1}{2 \prod_{i=1}^{3} \csc \alpha_i} \left\{ 3 \sum_{i=1}^{3} \csc^4 \alpha_i - \csc^4 \omega \right\}. \tag{14}$$

Now, from ((10); (iii)) we have

$$\{\csc^2 \omega\}^2 = \left\{\sum_{i=1}^{3} \csc^2 \alpha_i\right\}^2 \leqq 3 \sum_{i=1}^{3} \csc^4 \alpha_i . \tag{15}$$

Thus the right-hand side of (14) is positive; it is zero if and only if $\alpha_1 = \alpha_2 = \alpha_3$. It follows that the right-hand in (9) is greater than or equal to $3/\omega$ with equality if and only if $\alpha_1 = \alpha_2 = \alpha_3$. This finishes the proof of Theorem 1.
We end this note by remarking that a straightforward application of Holder's inequality on (7) gives

$$\frac{3}{\omega^\lambda} \leqq \sum_{i=1}^{3} \frac{1}{(\alpha_i - \omega)^\lambda} \tag{16}$$

for every $\lambda \geqq 1$.

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REFERENCES

1 Abi-Khuzam F.: Proof of Yff's Conjecture on the Brocard Angle of a Triangle. El. Math. *29*, 141–142 (1974).
2 Abi-Khuzam F.: Inequalities of Yff-type in the Triangle. El. Math. *35*, 80–81 (1980).
3 Abi-Khuzam F. and Boghossian A.: On Some Geometric Inequalities, to appear.
4 Johnson R: Advanced Euclidean Geometry. Dover Pbl., N.Y. (1960).
5 Mascioni V.: Zur Abschätzung des Brocardschen Winkels. El. Math. *41*, 98–101 (1986).

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On some inequalities connected with Fermat's equation

1. Introduction

In 1856 I. A. Grünert ([3], see also [6] p. 226) proved that if n is an integer, $n \geqq 2$ and $0 < x < y < z$ are real numbers satisfying the equation

$$x^n + y^n = z^n \tag{1}$$

then

$$z - y < \frac{x}{n}. \tag{2}$$

This result was rediscovered by G. Towes [7], and then by D. Zeitlin [8].