Star polytopes and the Schläfli function f(, ,)

Autor(en): **Coxeter, H.S.M.**

Objekttyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **44 (1989)**

Heft 2

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-41605

ELEMENTE DER MATHEMATIK

Revue de mathématiques élémentaires – Rivista di matematica elementare

Zeitschrift zur Pflege der Mathematik
und zur Förderung des mathematisch-physikalischen Unterrichts

El. Math. Vol. 44 Nr. 2 Seiten 25–56 Basel, März 1989

Star polytopes and the Schläfli function $f(\alpha, \beta, \gamma)$

The latter half of this article fills some gaps in Schläfli's very condensed treatment of the volume of a spherical simplex.

1. The Kepler-Poinsot polyhedra

For any rational number $\frac{n}{d} > 2$, there is a regular n-gon of density d, conveniently denoted by its Schläfli symbol $\{\frac{n}{d}\}$. For instance, $\{\frac{5}{2}\}$ is the *pentagram*, whose five sides surround its centre twice. Analogously in 3-space, for suitable rational numbers p and q, there is a regular polyhedron $\{p, q\}$ having face $\{p\}$ and vertex figure $\{q\}$. For instance, $\{\frac{5}{2}, 5\}$ is the *small stellated dodecahedron*, whose faces consist of twelve pentagrams, five at each of its twelve vertices. A famous mosaic, made in 1420 by Paolo Uccello [12, p. 20], is evidently intended to be a picture of this star polyhedron. $\{\frac{5}{2}, 5\}$ was rediscovered by J. Kepler, whose drawing of it is reproduced in Figure 1. He discovered also the *great stellated dodecahedron* $\{\frac{5}{2}, 3\}$, which has three pentagrams at each of its twenty vertices. L. Poinsot reciprocated the stellated dodecahedra to obtain the *great dodecahedron* $\{5, \frac{5}{2}\}$ and the *great icosahedron* $\{3, \frac{5}{2}\}$ [5, pp. 96, 114]. However, the former was actually drawn in 1568 by Jamnitzer [8a, Plate C.V.].

The planes of symmetry of any regular polyhedron $\{p, q\}$ decompose the concentric unit sphere into a pattern of spherical triangles [5, pp. 109–111]. If O is the centre, such a «characteristic triangle» ABC is determined by diameters OA, OB, OC which contain respectively a vertex, the midpoint of an incident edge, and the centre of an incident face. Therefore its angles are $A = \pi/q$, $B = \pi/2$, $C = \pi/p$ and its area is

$$\Delta_{p,q} = A + B + C - \pi = (2p + 2q - pq)\pi/2pq.$$

When p and q are integers, as they are for the five Platonic solids $\{3, 3\}$, $\{4, 3\}$, $\{3, 4\}$, $\{5, 3\}$, and $\{3, 5\}$, the characteristic triangle is a fundamental region for the symmetry group, whose order is accordingly

$$g = 4\pi/(A + B + C - \pi) = 8pq/(2p + 2q - pq).$$

This order is 120 for the icosahedron $\{3, 5\}$ and for the dodecahedron $\{5, 3\}$. In fact, their common symmetry group is

$$C_2 \times A_5.$$

The four Kepler-Poinsot polyhedra

$$\{\tfrac{5}{2},5\},\ \{5,\tfrac{5}{2}\},\ \{\tfrac{5}{2},3\},\ \{3,\tfrac{5}{2}\}$$

all have the same 15 planes of symmetry as those two «pentagonal» solids, but their 120 characteristic triangles, being larger, cover the whole sphere a number of times, say d times, where d is naturally called the *density* of the star polyhedron $\{p,q\}$. By comparing the areas [5, p. 111], we obtain

$$d = \frac{\Delta_{p,q}}{\Delta_{5,3}} = \frac{15}{pq}(2p+2q-pq), \tag{1.1}$$

which is

3 for $\{\tfrac{5}{2},5\}$, and $\{5,\tfrac{5}{2}\}$,

7 for $\{\tfrac{5}{2},3\}$, and $\{3,\tfrac{5}{2}\}$.

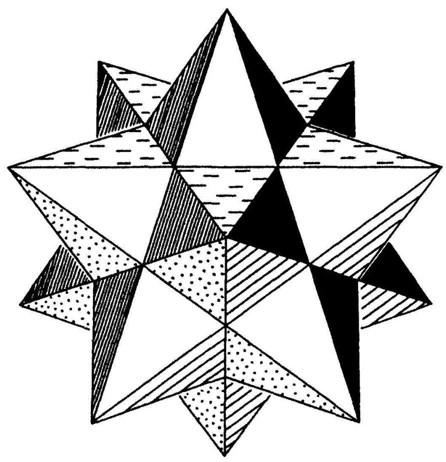

Figure 1

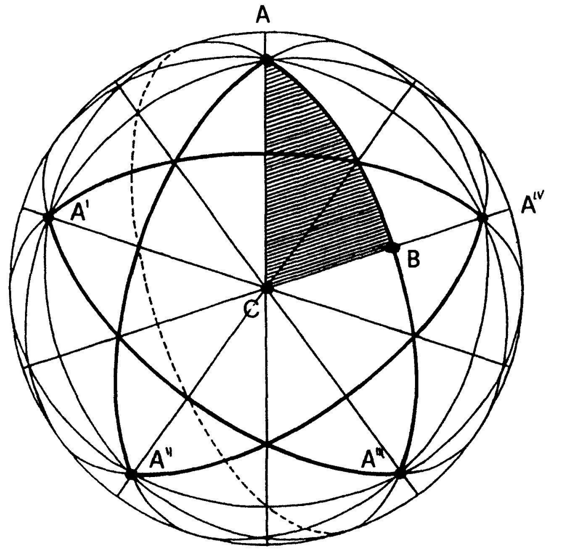

Figure 2

Figure 2, in which the shaded spherical triangle ABC is made up of three small triangles, illustrates the density 3 for $\{\tfrac{5}{2},5\}$. Mirrors AC and BC, inclined at $2\pi/5$, reflect the vertex A (where the angle is $\pi/5$) into the successive vertices of a pentagram $AA^{\mathrm{I}}A^{\mathrm{II}}A^{\mathrm{III}}A^{\mathrm{IV}}$ which is one face of $\{\tfrac{5}{2},5\}$, projected onto the circumsphere. The third mirror AB or AA^{I}, being one side of the pentagram, reflects this face into a neighbouring face.

2. The Schläfli-Hess polytopes

Analogously in 4-space, there is, for suitable rational numbers p, q, r, a regular polytope $\{p,q,r\}$ having facet (or cell) $\{p,q\}$ and vertex figure $\{q,r\}$. For instance [5, pp. 136, 191],

L. Schläfli discovered the six convex polytopes:

the 5-cell $\{3,3,3\}$, the 8-cell $\{4,3,3\}$, the 16-cell $\{3,3,4\}$,

the 24-cell $\{3,4,3\}$, the 120-cell $\{5,3,3\}$, the 600-cell $\{3,3,5\}$

[4, pp. 403, 404; 5, frontispiece].
Schläfli [10, pp. 297–298; 11, pp. 184–186, 267] discovered also four of the ten regular star-polytopes, namely two pairs of reciprocals:

the great stellated 120-cell	$\{\frac{5}{2},3,5\}$,
the grand 120-cell	$\{5,3,\frac{5}{2}\}$,
the grand 600-cell	$\{3,3,\frac{5}{2}\}$,
the great grand stellated 120-cell	$\{\frac{5}{2},3,3\}$,

[5, p. 294; 6, p. 46]. These four were rediscovered about thirty years later by E. Hess [8], who added

the stellated 120-cell	$\{\frac{5}{2},5,3\}$,
the icosahedral 120-cell	$\{3,5,\frac{5}{2}\}$,
the great 120-cell	$\{5,\frac{5}{2},5\}$,
the grand stellated 120-cell	$\{\frac{5}{2},5,\frac{5}{2}\}$,
the great grand 120-cell	$\{5,\frac{5}{2},3\}$,
the great icosahedral 120-cell	$\{3,\frac{5}{2},5\}$.

Schläfli failed to recognize these six because, although he had rediscovered $\{\frac{5}{2},3\}$ and $\{3,\frac{5}{2}\}$, it seems that nobody had ever shown him a model of $\{\frac{5}{2},5\}$ or $\{5,\frac{5}{2}\}$ (which have 12 vertices, 30 edges and 12 faces) and he believed Euler's formula to be necessary for the existence of a polyhedron; accordingly the symbol $\{p,q,r\}$ for a polytope could not admit the numbers 5 and $\frac{5}{2}$ side by side!
The hyperplanes of symmetry of any regular polytope $\{p,q,r\}$ decompose the concentric unit 3-sphere into a pattern of spherical orthoschemes [5, pp. 130, 137]. If O is the centre, such a «characteristic orthoscheme» $ABCD$ is determined by diameters OA, OB, OC, OD which contain respectively a vertex, the midpoint of an incident edge, the centre of an incident face, and the centre of an incident facet. Therefore its edges AB, BC, CD are mutually perpendicular, and the dihedral angles along its six edges CD, AD, AB, AC, BC, BD are

$$(CD)=\pi/p,\ (AD)=\pi/q,\ (AB)=\pi/r,\ (AC)=(BC)=(BD)=\pi/2.$$

When p,q,r are integers, as they are for the six convex polytopes, the characteristic orthoscheme is a fundamental region for the symmetry group, whose order is the ratio of the volumes of the whole 3-sphere and the orthoscheme. This order [5, p. 153] is

$$120^2 = 14\,400$$

for the 120-cell $\{5, 3, 3\}$ and for the 600-cell $\{3, 3, 5\}$, and also for all the ten star polytopes, which share the same sixty hyperplanes of symmetry [5, p. 266]. The 14400 characteristic orthoschemes of the star polytopes, being larger than those of the 120-cell, cover the 3-sphere a number of times, say d times, where d is naturally called the *density* of the star-polytope $\{p, q, r\}$. Thus d (Schläfli's h) can computed as the ratio of the volumes of the characteristic orthoschemes for $\{p, q, r\}$ and $\{5, 3, 3\}$.

To carry out this computation, we investigate, as Lobachevsky and Schläfli did, the volume of the general 3-dimensional orthoscheme [7, § 2], whose dihedral angles are

$$(CD) = \alpha, \ (AD) = \beta, \ (AB) = \gamma, \ (AC) = (BC) = (BD) = \pi/2,$$

as in Figure 3.

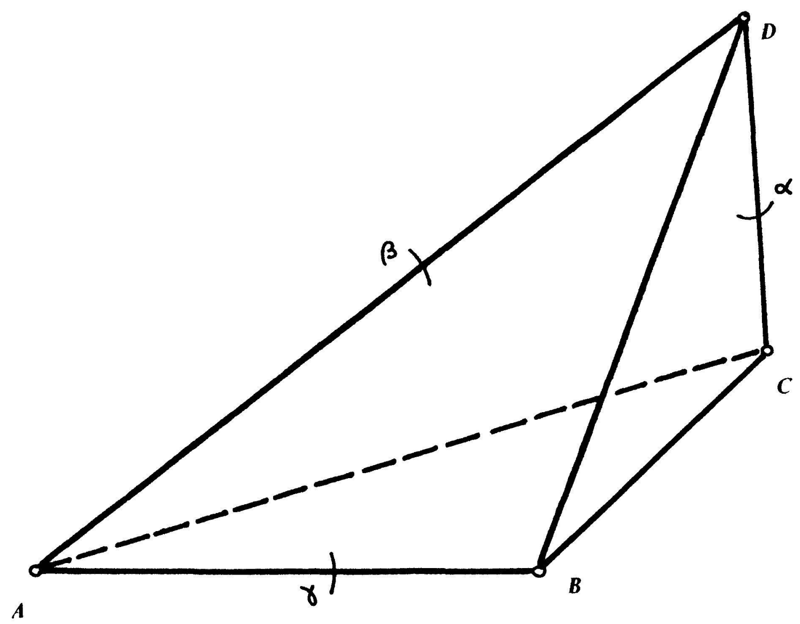

Figure 3

3. The 3-dimensional orthoscheme

In any kind of 3-space, the shape of a tetrahedron is determined by its six dihedral angles. If the space is spherical (or elliptic, or hyperbolic), these dihedral angles determine its shape and size. In particular, if the consecutive edges AB, BC, CD are mutually orthogonal, the tetrahedron is an *orthoscheme* (see Figure 3). It is determined by the angles α, β, γ along the edges CD, AD, AB, while the remaining edges AC, BC, BD have right dihedral angles. Let us simply call it

$$(\alpha, \beta, \gamma),$$

with the understanding that it could equally well be called (γ, β, α) [10, p. 258; 11, p. 248]. If the 3-space is spherical (or elliptic), the dihedral angles satisfy

$$\sin\alpha \sin\gamma > \cos\beta.$$

The edges and face-angles can be computed with the help of a triangular scheme of numbers

$$\begin{array}{c} (-1,-1)\quad (0,0)\quad (1,1)\quad (2,2)\quad (3,3)\quad (4,4) \\ (-1,0)\quad (0,1)\quad (1,2)\quad (2,3)\quad (3,4) \\ (-1,1)\quad (0,2)\quad (1,3)\quad (2,4) \\ (-1,2)\quad (0,3)\quad (1,4) \\ (-1,3)\quad (0,4) \\ (-1,4) \end{array}$$

where $(s,s)=0$, $(s,s+1)=1$; one of the numbers $(s-1,s+1)$ in the third row can be given any convenient positive value and the rest of them are then determined by the equations

$$(-1,1)\,(0,2)=\sec^2\alpha,\quad (0,2)\,(1,3)=\sec^2\beta,\quad (1,3)\,(2,4)=\sec^2\gamma. \tag{3.1}$$

The remaining rows are given by the formula

$$(s,t)=\frac{(s,t-1)\,(s+1,t)-1}{(s+1,t-1)} \tag{3.2}$$

which is a consequence of the symmetrical rule

$$(t,u)\,(s,v)+(u,s)\,(t,v)+(s,t)\,(u,v)=0$$

[5, p. 160; 3, p. 204; 6, p. 56].

All the trigonometric functions of the angles and edges of the spherical orthoscheme are very simply expressible in terms of these two-digit symbols. In particular, the edges

$$a=CD,\quad b=AD,\quad c=AB,$$

which carry the dihedral angles α, β, γ, are given by

$$\sec^2 a=\frac{(-1,3)\,(2,4)}{(-1,2)},\quad \sec^2 b=(-1,3)\,(0,4),\quad \sec^2 c=\frac{(-1,1)\,(0,4)}{(1,4)} \tag{3.3}$$

or

$$\tan^2 a=\frac{(-1,4)}{(-1,2)},\qquad \tan^2 b=\frac{(-1,4)}{(0,3)},\qquad \tan^2 c=\frac{(-1,4)}{(1,4)}. \tag{3.4}$$

The limiting case when the spherical orthoscheme becomes Euclidean is given by

$$(-1,4)=0, \tag{3.5}$$

which implies $a=b=c=0$.

The one degree of freedom in our choice of the numbers $(s-1, s+1)$ may naturally be used to split the product $(0,2)(1,3)$, so that

$$(0,2) = (1,3) = \sec\beta.$$

Then, by (3.1),

$$(-1,1) = \sec^2\alpha\cos\beta, \quad (2,4) = \cos\beta\sec^2\gamma,$$

and by (3.2),

$$(-1,2) = \tan^2\alpha, \quad (0,3) = \tan^2\beta, \quad (1,4) = \tan^2\gamma,$$

$$(-1,3) = \frac{\sin^2\alpha - \cos^2\beta}{\cos^2\alpha\cos\beta}, \quad (0,4) = \frac{\sin^2\gamma - \cos^2\beta}{\cos^2\gamma\cos\beta},$$

$$(-1,4) = \frac{\sin^2\alpha\sin^2\gamma - \cos^2\beta}{\cos^2\alpha\cos^2\gamma}. \tag{3.6}$$

Using this notation, we have

$$\cos a = \frac{\sin\alpha\cos\gamma}{\sqrt{\sin^2\alpha - \cos^2\beta}}, \quad \cos b = \frac{\cos\alpha\cos\beta\cos\gamma}{\sqrt{\sin^2\alpha - \cos^2\beta}\sqrt{\sin^2\gamma - \cos^2\beta}},$$

$$\cos c = \frac{\sin\gamma\cos\alpha}{\sqrt{\sin^2\gamma - \cos^2\beta}} \tag{3.7}$$

[11, p. 156]. Also, by (3.5) and (3.6), $a = b = c = 0$ when

$$\sin\alpha\sin\gamma = \cos\beta. \tag{3.8}$$

4. The Schläfli function $f(\alpha, \beta, \gamma)$

Schläfli [10, p. 235; 11, p. 167, 234; 2, pp. 285–288] investigated the volume S of the general spherical tetrahedron, taking as his unit the volume of the *orthant* $(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi)$ (that is, one-sixteenth of the whole 3-sphere). He showed that S satisfies the differential equation

$$dS = \frac{4}{\pi^2}\sum l\,d\lambda,$$

where the summation is over the six edges l which carry the dihedral angles λ.

In the case of the orthoscheme (α, β, γ), with volume $f(\alpha, \beta, \gamma)$, this becomes

$$d f(\alpha, \beta, \gamma) = \frac{4}{\pi^2}(a\, d\alpha + b\, d\beta + c\, d\gamma). \tag{4.1}$$

(Schläfli, in his earlier paper [11, pp. 156–163], used a different unit of measurement and thus eliminated the coefficient $4/\pi^2$.)
In the simple case when $\beta = \frac{1}{2}\pi$, we have a spherical ‹disphenoid› $(\alpha, \frac{1}{2}\pi, \gamma)$ which joins arcs $AB = c = \alpha$ and $CD = a = \gamma$ of two polar great circles, so we can expect the volume to be a numerical multiple of the product $ca = \alpha\gamma$. In fact,

$$df(\alpha, \tfrac{1}{2}\pi, \gamma) = \frac{4}{\pi^2}(\gamma\, d\alpha + \alpha\, d\gamma) = \frac{4}{\pi^2} d(\alpha\gamma)$$

and, since $f(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi) = 1$,

$$f(\alpha\, \tfrac{1}{2}\pi, \gamma) = \frac{4}{\pi^2}\alpha\gamma. \tag{4.2}$$

When one of α, β, γ is changed to its supplement, $f(\alpha, \beta, \gamma)$ changes in an easily described manner, suggested by another part of Schläfli's work [10, p. 240; 11, p. 238]. The three rays issuing from A along the edges AB, AC, AD of the orthoscheme $S = ABCD$ (see Figure 3) will meet again in the antipodes A' of A and determine another orthoscheme $S_1 = A'BCD$, of type $(\pi - \alpha, \beta, \gamma)$. The trihedron $S \cup S_1$ intersects the equatorial 2-sphere of the poles A and A' in a spherical triangle with angles $\beta, \gamma, \frac{1}{2}\pi$ (these being the dihedral angles along the edges AD, AB, AC). The volume of the trihedron is the same fraction of the whole 3-sphere as the area of the spherical triangle is of the equatorial 2-sphere, so it amounts to $16(\beta + \gamma - \frac{1}{2}\pi)/4\pi$ orthants:

$$f(\alpha, \beta, \gamma) + f(\pi - \alpha, \beta, \gamma) = \frac{4}{\pi}(\beta + \gamma) - 2 \tag{4.3}$$

and, in particular,

$$f(\tfrac{1}{2}\pi, \beta, \gamma) = \frac{2}{\pi}(\beta + \gamma) - 1. \tag{4.4}$$

Similarly, the three rays issuing from B along the edges BA, BC, BD of S will meet again is the antipodes B' of B and determine another orthoscheme $S_2 = AB'CD$, of type $(\pi - \alpha, \pi - \beta, \gamma)$. The same reasoning as before, applied to the trihedron $S \cup S_2$, gives

$$f(\alpha, \beta, \gamma) + f(\pi - \alpha, \pi - \beta, \gamma) = \frac{4}{\pi}\gamma.$$

When α is replaced by $\pi - \alpha$, this combines with (4.3) to give

$$f(\alpha, \beta, \gamma) - f(\alpha, \pi - \beta, \gamma) = \frac{4\beta}{\pi} - 2. \tag{4.5}$$

Schläfli found also some remarkable connections between $(\alpha, \pi/3, \pi/3)$, $(2\alpha, \alpha, \pi/3)$ and $(\alpha, 2\alpha, \alpha)$.

If $\beta = \gamma = \pi/3$, (3.7) yields

$$\cos a = \frac{\sin\alpha}{\sqrt{4\sin^2\alpha - 1}}. \tag{4.6}$$

By (4.1), for this value of a,

$$d f\left(\alpha, \frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{4}{\pi^2} a\, d\alpha. \tag{4.7}$$

If, instead, $\beta = 2\alpha$ and $\gamma = \alpha$, (3.7) yields

$$\cos a = \frac{\sin\alpha\cos\alpha}{\sqrt{\sin^2 2\alpha - \cos^2\alpha}} = \frac{\sin\alpha}{\sqrt{4\sin^2\alpha - 1}}$$

as in (4.6), and

$$\cos b = \frac{\cos^2\alpha\cos 2\alpha}{\sin^2 2\alpha - \cos^2\alpha} = \frac{\cos 2\alpha}{4\sin^2\alpha - 1} = \frac{2\sin^2\alpha}{4\sin^2\alpha - 1} - 1 = 2\cos^2 a - 1 = \cos 2a,$$

whence $b = 2a$. Since $\gamma = \alpha$, we have also $c = a$, and

$$d f(\alpha, 2\alpha, \alpha) = \frac{4}{\pi^2}\{a\, d\alpha + 2a\, d(2\alpha) + a\, d\alpha\} = \frac{4}{\pi^2}\cdot 6a\, d\alpha = 6\, d f(\alpha, \pi/3, \pi/3).$$

Thus

$$f(\alpha, 2\alpha, \alpha) = 6 f(\alpha, \pi/3, \pi/3) \tag{4.8}$$

[11, p. 161 (21)]. No constant needs to be added since, by (3.8), both sides of the equation (4.8) become zero when $3\sin^2\alpha = 1$.

In the case of $f(2\alpha, \alpha, \pi/3)$, we must replace, in (3.7), α by 2α, β by α, and γ by $\pi/3$ obtaining

$$\cos a = \frac{\sin 2\alpha}{\sqrt{4\sin^2 2\alpha - 4\cos^2\alpha}} = \frac{\sin\alpha}{\sqrt{4\sin^2\alpha - 1}}$$

as in (4.6), and

$$\cos b = \frac{\cos 2\alpha\cos\alpha}{\sqrt{\sin^2 2\alpha - \cos^2\alpha}\sqrt{4\sin^2\alpha - 1}} = \frac{\cos 2\alpha}{4\sin^2\alpha - 1},$$

whence $b = 2a$ again, and

$$d f(2\alpha, \alpha, \pi/3) = a\, d(2\alpha) + 2a\, d\alpha = 4a\, d\alpha = 4\, d f(\alpha, \pi/3\ \pi/3).$$

Thus

$$f(2\alpha, \alpha, \pi/3) = 4f(\alpha, \pi/3, \pi/3). \tag{4.9}$$

[10, p. 266; 11, p. 160(17)]. Again both sides vanish when $\sqrt{3}\sin\alpha = 1$.
Equations (4.8) and (4.9) are two cases of an n-dimensional result [10, p. 267; 11, p. 256] which is relevant to the theory of ‹Minowskian star polytopes› [1, p. 22] or ‹hyperbolic star honeycombs› [3, p. 210]. H.E. Debrunner [7] has discovered a geometric approach to these formulae. The three-dimensional case may be described as follows.
In Euclidean or non-Euclidean 3-space, a regular tetrahedron with dihedral angle 2α is decomposed by its six planes of symmetry into $4! = 24$ orthoschemes $(\alpha, \frac{\pi}{3}, \frac{\pi}{3})$. Of these six planes, two join one edge of the regular tetrahedron to the midpoint of the opposite edge, and *vice versa*, decomposing the tetrahedron into 4 orthoschemes $(\alpha, 2\alpha, \alpha)$. On the other hand, three of the six planes join one vertex to the three medians of the opposite face, decomposing the tetrahedron into 6 orthoschemes $(2\alpha, \alpha, \frac{\pi}{3})$. Since $24 = 4 \times 6$, each $(\alpha, 2\alpha, \alpha)$ can be decomposed into $6(\alpha, \frac{\pi}{3}, \frac{\pi}{3})$'s, and each $(2\alpha, \alpha, \frac{\pi}{3})$ into 4.

5. Rational volumes

Although Schläfli's equation (4.1) cannot be integrated in terms of elementary functions, something remarkable happens when

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{5.1}$$

[10, p. 263; 11, pp. 159, 175, 260]. In such cases (3.7) becomes

$$a = \tfrac{1}{2}\pi - \alpha, \quad b = \beta, \quad c = \tfrac{1}{2}\pi - \gamma$$

and thus (4.1) becomes

$$\begin{aligned} d f(\alpha, \beta, \gamma) &= \frac{4}{\pi^2}\left\{\left(\frac{1}{2}\pi - \alpha\right) d\alpha + \beta\, d\beta + \left(\frac{1}{2}\pi - \gamma\right) d\gamma\right\} \\ &= \frac{2\beta}{\pi} d\left(\frac{2\beta}{\pi}\right) - \left(1 - \frac{2\alpha}{\pi}\right) d\left(1 - \frac{2\alpha}{\pi}\right) - \left(1 - \frac{2\gamma}{\pi}\right) d\left(1 - \frac{2\gamma}{\pi}\right). \end{aligned}$$

Therefore, when (5.1) holds,

$$f(\alpha, \beta, \gamma) = \frac{1}{2}\left\{\left(\frac{2\beta}{\pi}\right)^2 - \left(1 - \frac{2\alpha}{\pi}\right)^2 - \left(1 - \frac{2\gamma}{\pi}\right)^2\right\}.$$

No constant needs to be added, since if $\alpha = \beta + \gamma = \dfrac{1}{2}\pi$, (4.4) yields

$$f\left(\frac{1}{2}\pi, \beta, \gamma\right) = \frac{2}{\pi}(\beta + \gamma) - 1 = 0 = \frac{1}{2}\left\{\left(\frac{2\beta}{\pi}\right)^2 - 0 - \left(\frac{2\beta}{\pi}\right)^2\right\}.$$

We know [5, p. 109] that the only solutions of (5.1), in terms of acute angles commensurable with π, are the three permutations of $\pi/4$, $\pi/3$, $\pi/3$ and the six permutations of $2\pi/5$, $\pi/5$, $\pi/3$ [10, pp. 268–269; 11, pp. 159–160, 180–181]. These yield the five special volumes

$$f\left(\frac{\pi}{4},\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{1}{24},\quad f\left(\frac{\pi}{3},\frac{\pi}{4},\frac{\pi}{3}\right)=\frac{1}{72},\quad f\left(\frac{2\pi}{5},\frac{\pi}{5},\frac{\pi}{3}\right)=\frac{1}{225},$$
$$f\left(\frac{2\pi}{5},\frac{\pi}{3},\frac{\pi}{5}\right)=\frac{1}{45},\quad f\left(\frac{\pi}{5},\frac{2\pi}{5},\frac{\pi}{3}\right)=\frac{19}{225}. \tag{5.2}$$

By (4.3) and (4.9),

$$f\left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right)+f\left(\frac{2\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{4}{\pi}\left(\frac{2\pi}{3}-\frac{\pi}{2}\right)=\frac{2}{3}\text{ and } f\left(\frac{2\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right)=4f\left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right);$$

therefore

$$f\left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{2}{15}. \tag{5.3}$$

By (4.9),

$$f\left(\frac{\pi}{5},\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{1}{4}f\left(\frac{2\pi}{5},\frac{\pi}{5},\frac{\pi}{3}\right)=\frac{1}{900},$$

and by (4.8),

$$f\left(\frac{\pi}{5},\frac{2\pi}{5},\frac{\pi}{5}\right)=6f\left(\frac{\pi}{5},\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{1}{150}.$$

By (4.3) and (5.2),

$$f\left(\frac{4\pi}{5},\frac{2\pi}{5},\frac{\pi}{3}\right)=\frac{4}{\pi}\left(\frac{2\pi}{5}+\frac{\pi}{3}-\frac{\pi}{2}\right)-f\left(\frac{\pi}{5},\frac{2\pi}{5},\frac{\pi}{3}\right)=\frac{14}{15}-\frac{19}{225}=\frac{191}{225},$$

whence, by (4.9) again,

$$f\left(\frac{2\pi}{5},\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{191}{900} \tag{5.4}$$

[10, pp. 268–269 (9); 11, pp. 160 (20), 181 (19)]. By (4.5) and (4.8),

$$f\left(\frac{2\pi}{5},\frac{\pi}{5},\frac{2\pi}{5}\right)=f\left(\frac{2\pi}{5},\frac{4\pi}{5},\frac{2\pi}{5}\right)+\frac{4}{5}-2$$

$$=6f\left(\frac{2\pi}{5},\frac{\pi}{3},\frac{\pi}{3}\right)-\frac{6}{5}=\frac{191}{150}-\frac{6}{5}=\frac{11}{150} \qquad (5.5)$$

[10, p. 269 (11); 11, pp. 162 (23), 181 (21)].
Also, by (3.8), there is one trivially rational volume:

$$f\left(\frac{\pi}{4},\frac{\pi}{3},\frac{\pi}{4}\right)=0. \qquad (5.6)$$

And of course there are infinitely many rational instances of (4.2) and (4.4).

6. The ten star polytopes and their densities

We can now use Schläfli's formula

$$d=\frac{f(\pi/p,\pi/q,\pi/r)}{f(\pi/5,\pi/3,\pi/3)}=900f\left(\frac{\pi}{p},\frac{\pi}{q},\frac{\pi}{r}\right)$$

[11, p. 186; p. 287] to do, for all the ten star polytopes, what he did for his four, namely, to compute their densities [1; 5, pp. 284, 294]:

4 for $\{\frac{5}{2},5,3\}$ and $\{3,5,\frac{5}{2}\}$,
6 for $\{5,\frac{5}{2},5\}$,
20 for $\{\frac{5}{2},3,5\}$ and $\{5,3,\frac{5}{2}\}$,
66 for $\{\frac{5}{2},5,\frac{5}{2}\}$,
76 for $\{5,\frac{5}{2},3\}$ and $\{3,\frac{5}{2},5\}$,
191 for $\{\frac{5}{2},3,3\}$ and $\{3,3,\frac{5}{2}\}$.

After mentioning the honeycombs $\{4,3,3,4\}$, $\{3,3,4,3\}$, $\{3,4,3,3\}$ [5, p. 153] which tessellate Euclidean 4-space, Schläfli [11, p. 187] unhappily added two «star honeycombs», $\{5,3,3,\frac{5}{2}\}$ and $\{\frac{5}{2},3,3,5\}$, which he declared to have density 191, whereas their density is really infinite [1, pp. 509, 521; 5, pp. 264, 285]. This is analogous to an attempt to tessellate the Euclidean plane with pentagons so as to obtain a «tessellation» $\{5,\frac{10}{3}\}$ [4, p. 63].

H. S. M. Coxeter, Department of Mathematics, University of Toronto

REFERENCES

1 Coxeter H. S. M.: The densities of the regular polytopes. Proc. Camb. Phil. Soc. *27*, 201–211, (1931); *28*, 509–521 (1932); *29*, 1–22 (1933).
2 Coxeter H. S. M.: Non-Euclidean Geometry, 5th ed., University of Toronto Press, Toronto, 1965.
3 Coxeter H. S. M.: Twelve Geometric Essays. Southern Illinois University Press, Carbondale, 1968.
4 Coxeter H. S. M.: Introduction to Geometry, 2nd ed., Wiley, New York, 1969.
5 Coxeter H. S. M.: Regular Polytopes, 3rd ed., Dover, New York, 1973.
6 Coxeter H. S. M.: Regular Complex Polytopes, Cambridge University Press, 1974.
7 Debrunner H. E.: Decomposing orthoschemes into orthoschemes, to appear.
8 Hess E.: Über die regulären Polytope höherer Art. Sitzungsber. Gesells. Beförderung gesammten Naturwiss. Marburg, 1885, 31–57.
8a Jamnitzer W.: Perspectiva corporum regularium. Verlag Biermann und Boukes, Frankfurt, 1972. See also El. Math. *11*, 97–100, (1956).
9 Lobachevsky N. I.: Imaginäre Geometrie und Anwendung der imaginären Geometrie auf einige Integrale. Teubner, Leipzig, 1904.
10 Schläfli L.: Gesammelte Mathematische Abhandlungen I. Birkhäuser, Basel, 1950.
11 Schläfli L.: Gesammelte Mathematische Abhandlungen II. Birkhäuser, Basel, 1953.
12 Wills J. M.: Mathematik und Kunst: Die platonischen Körper. Siegner Studien *35*, 17–27, (1983/84).

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Eine Anwendung des Gaußschen Integralsatzes

Mittels einer geeigneten komplexen Formulierung des Gaußschen Integralsatzes für die Ebene werden Formeln für die Trägheitsmomente [1*] eines Dreiecks hergeleitet, die in einfacher Weise lediglich von den Eckpunkten abhängen.
Mit G werde ein beschränktes Gebiet in der x, y-Ebene bezeichnet, dessen Rand aus einer einfach geschlossenen positiv orientierten stückweise stetig differenzierbaren Kurve ∂G besteht. Für ein stetig differenzierbares Vektorfeld

$$(u, v)\colon\ \bar{G} \to \mathbb{R}^2$$

gilt der Gaußsche Satz:

$$\iint_G \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\, dy = \int_{\partial G} (u\, dy - v\, dx),$$

und entsprechend erhält man für das Vektorfeld $(v, -u)$

$$\iint_G \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy = \int_{\partial G} (u\, dx + v\, dy).$$