# A remark on the gamma function


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Objekttyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **44 (1989)**

Heft 3

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-41612

| | | | | | Number of primes given by $an^3+bn^2+cn+d$ for | | | | |
|---|---|---|---|---|---|---|---|---|---|
| No | a | b | c | d | $n<100$ | $n<200$ | $n<300$ | $n<400$ | $n<500$ |
| (1) | 1 | −220 | 16119 | −392723 | 75 | 134 | 179 | 219 | 261 |
| (2) | 1 | −199 | 13190 | −290869 | 75 | 124 | 163 | 206 | 235 |
| (3) | 1 | −160 | 8547 | −142811 | 75 | 130 | 179 | 221 | 264 |
| (4) | 1 | −159 | 8420 | −148153 | 76 | 124 | 164 | 203 | 238 |
| (5) | 1 | −151 | 7610 | −129097 | 76 | 125 | 168 | 204 | 245 |
| (6) | 1 | −150 | 7493 | −124277 | 76 | 128 | 170 | 217 | 250 |
| (7) | 1 | −137 | 6270 | −95203 | 75 | 121 | 168 | 197 | 233 |
| (8) | 1 | −125 | 5196 | −73291 | 79 | 130 | 174 | 212 | 254 |
| (9) | 1 | −119 | 4718 | −71741 | 75 | 118 | 158 | 190 | 224 |
| (10) | 1 | −114 | 4343 | −54829 | 76 | 125 | 171 | 200 | 232 |
| (11) | 1 | −111 | 4100 | −49367 | 76 | 119 | 150 | 199 | 246 |
| (12) | 1 | −97 | 3126 | −32603 | 75 | 115 | 161 | 195 | 235 |
| (13) | 1 | −96 | 3059 | −32563 | 75 | 127 | 162 | 192 | 225 |
| (14) | 1 | −82 | 2237 | −20407 | 75 | 112 | 149 | 183 | 218 |
| (15) | 2 | −489 | 39847 | −1084553 | 75 | 134 | 176 | 222 | 267 |
| (16) | 2 | −372 | 23050 | −476027 | 75 | 128 | 174 | 211 | 239 |
| (17) | 2 | −292 | 14202 | −231551 | 76 | 124 | 160 | 206 | 252 |
| (18) | 2 | −289 | 13917 | −221891 | 75 | 124 | 172 | 210 | 247 |
| (19) | 2 | −281 | 13157 | −204487 | 78 | 129 | 169 | 214 | 250 |
| (20) | 2 | −280 | 13072 | −203857 | 76 | 123 | 168 | 204 | 240 |
| (21) | 2 | −199 | 6595 | −79657 | 79 | 125 | 174 | 217 | 257 |

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0013-6018/89/030070-04$1.50+0.20/0

# A remark on the gamma function

According to Problem 188, Part II of G. Pólya and G. Szegő [2] for each integrable function $f(x)$ on $0 \leqq x \leqq 1$ we have:

$$\lim_{n\to\infty} \frac{1}{\varphi(n)} \sum_{\substack{k=1 \\ (k,n)=1}}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx,$$

or

$$\sum_{\substack{k=1\\(k,n)=1}}^{n} f\left(\frac{k}{n}\right) \sim \varphi(n) \int_0^1 f(x)\,dx \quad (n \to \infty),$$

where $\varphi(n)$ denotes Euler's totient function.

Let $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt$ be the Euler gamma function. Then using the above mentioned result for $f(x) = \log \Gamma(x)$ and taking into account that $\int_0^1 \log \Gamma(x)\,dx = \log \sqrt{2\pi}$ (Raabe's integral; [3]) we obtain

$$\sum_{\substack{k=1\\(k,n)=1}}^{n} \log \Gamma\left(\frac{k}{n}\right) \sim \varphi(n) \log \sqrt{2\pi},$$

or

$$\log P(n) \sim \varphi(n) \log \sqrt{2\pi}, \quad \text{where} \quad P(n) \equiv \prod_{\substack{k=1\\(k,n)=1}}^{n} \Gamma\left(\frac{k}{n}\right).$$

The aim of this note is to give an explicit formula for $P(n)$ and to establish an asymptotic formula with remainder term for the summatory function of $\log P(n)$.

We shall use the following results:

**Lemma 1** ([3]).

$$\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}, \quad n > 1.$$

**Lemma 2.** (A. Hurwitz, see Problem 35, Part VIII of [2])

*Suppose $\psi(x)$ is an arbitrary function defined for $0 \leqq x \leqq 1$.*

*If $f(n) = \sum\limits_{\substack{k=1\\(k,n)=1}}^{n} \psi\left(\frac{k}{n}\right)$ and $g(n) = \sum\limits_{k=1}^{n} \psi\left(\frac{k}{n}\right)$, then $f(n) = \sum\limits_{d \mid n} \mu(d)\, g\left(\frac{n}{d}\right)$, $\mu$ denoting the Möbius function.*

**Lemma 3.**

*If $F(n) = \prod\limits_{\substack{k=1\\(k,n)=1}}^{n} \psi^*\left(\frac{k}{n}\right)$, $G(n) = \prod\limits_{k=1}^{n} \psi^*\left(\frac{k}{n}\right)$ and $\psi^*(x) > 0$ for every $x$, $0 \leqq x \leqq 1$, then*

$$F(n) = \sum_{d \mid n} [G(d)]^{\mu\left(\frac{n}{d}\right)}.$$

*Proof:* Apply Lemma 2 for $f=\log F$, $g=\log G$, $\psi=\log\psi^*$.

**Theorem 1.**

*For $n>1$ we have*

$$P(n)=\frac{(2\pi)^{\varphi(n)/2}}{\exp \Lambda(n)/2}=\begin{cases}(2\pi)^{\varphi(n)/2}/\sqrt{p}, & \text{for} \quad n=p^m\\ (2\pi)^{\varphi(n)/2}, & \text{for} \quad n\neq p^m\end{cases},$$

*where $\Lambda(n)$ is von Mangoldt's arithmetic function.*

*Proof*: Using Lemma 1 und Lemma 3 one obtains successively:

$$P(n)=\prod_{d\mid n}\left[\frac{(2\pi)^{(d-1)/2}}{\sqrt{d}}\right]^{\mu\left(\frac{n}{d}\right)}=(2\pi)^{\frac{1}{2}\sum_{d\mid n} d\mu\left(\frac{n}{d}\right)-\frac{1}{2}\sum_{d\mid n}\mu\left(\frac{n}{d}\right)}\Big/\sqrt{h(n)},$$

where

$$\sum_{d\mid n} d\mu\left(\frac{n}{d}\right)=\varphi(n);\ \sum_{d\mid n}\mu\left(\frac{n}{d}\right)=0\ (n>1);\ \text{and}\ h(n)\equiv\prod_{d\mid n} d^{\mu\left(\frac{n}{d}\right)}.$$

Here

$$\log h(n)=\sum_{d\mid n}\mu\left(\frac{n}{d}\right)\log d=\Lambda(n),$$

cf. [1], and the proof is complete.

**Lemma 4** (cf. [1]).

$$\sum_{n\leqq x}\varphi(n)=\frac{3}{\pi^2}x^2+O(x\log x),$$

$$\sum_{n\leqq x}\Lambda(n)=O(x).$$

**Theorem 2.**

$$\sum_{n\leqq x}\log P(n)=\frac{3\log 2\pi}{2\pi^2}x^2+O(x\log x).$$

*Proof:* By Theorem 1 and Lemma 4:

$$\sum_{n \leqq x} \log P(n) = \sum_{n \leqq x} \left[ \frac{\varphi(n)}{2} \log 2\pi - \tfrac{1}{2} \Lambda(n) \right] =$$

$$= \frac{\log 2\pi}{2} \sum_{n \leqq x} \varphi(n) - \tfrac{1}{2} \sum_{n \leqq x} \Lambda(n) =$$

$$= \frac{\log 2\pi}{2} \left( \frac{3}{\pi^2} x^2 + O(x \log x) \right) - \tfrac{1}{2} O(x) =$$

$$= \frac{3 \log 2\pi}{2\ \pi^2} x^2 + O(x \log x).$$

The authors wish to thank the referee for valuable suggestions.

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 0013-6018/89/030073-04$1.50 + 0.20/0

# Kleine Mitteilungen

## Zu einer Aufgabe der Kombinatorik

$A_p^k$; $p \in \mathbb{N}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ sei die Anzahl der Möglichkeiten, $k$ (nicht zu unterscheidende) Dinge auf $p$ (zu unterscheidende) Personen aufzuteilen. Das Resultat

$$A_p^k = \binom{p+k-1}{k}; \quad p \in \mathbb{N}, k \in \mathbb{N}_0 \tag{1}$$

wird im allgemeinen durch vollständige Induktion gewonnen, wobei

$$A_p^2 = \binom{p+2-1}{2}, \quad A_p^3 = \binom{p+3-1}{3}$$