On a discrete Dido-type question

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Der Satz über die Division mit Rest läßt sich auf die Partialbruchzerlegung anwenden. Hat man in der Produktdarstellung (4) quadratische Faktoren $q(s) = s^2 + \beta s + \gamma$ und ist dort etwa $Q_1 = q^m$, dann läßt sich der zugehörige Partialbruch $\frac{P_1}{q^m}$ in (5) mit $\deg P_1 < 2m$ durch Division mit Rest in die bekannte Form

$$\frac{P_1(s)}{(q(s))^m} = \frac{b_m s + c_m}{(q(s))^m} + \frac{b_{m-1} s + c_{m-1}}{(q(s))^{m-1}} + \ldots + \frac{b_1 s + c_1}{q(s)} \tag{8}$$

mit eindeutig bestimmten Konstanten $b_1, c_1, \ldots, b_m, c_m$ bringen. Das Verfahren wird am folgenden Beispiel erläutert.

Beispiel. Ist etwa $P_1(s) = 2s^5 - s^4 + 4s^3 + 1$ und $Q_1(s) = (s^2+1)^3$, so führt wiederholte Division durch $s^2 + 1$ auf

$$2s^5 - s^4 + 4s^3 + 1 = (2s^3 - s^2 + 2s + 1)(s^2+1) - 2s,$$

$$2s^3 - s^2 + 2s + 1 = (2s-1)(s^2+1) + 2,$$

und damit lautet die Partialbruchzerlegung (8) in diesem Fall

$$\frac{2s^5 - s^4 + 4s^3 + 1}{(s^2+1)^3} = -\frac{2s}{(s^2+1)^3} + \frac{2}{(s^2+1)^2} + \frac{2s-1}{s^2+1}.$$

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On a discrete Dido-type question

We start with the following well-known fact [1]. If D is a simply connected domain of the Euclidean plane with area $\mathscr{A}(D)$ whose boundary is divided into a segment and a simple curve Γ of length $L(\Gamma)$, then $\mathscr{A}(D) \leqq \frac{1}{2\cdot\pi} \cdot L^2(\Gamma)$ with equality if and only if D is a hemicircle. In other words if we have a simple curve Γ of given length $L(\Gamma)$ in the Euclidean plane, then the area of its convex hull is maximal if and only if Γ is a hemicircle i.e. $\mathscr{A}(\operatorname{conv}\Gamma) \leqq \frac{1}{2\cdot\pi} \cdot L^2(\Gamma)$. Reading these sentences we immediately thought of the following discrete version of the above problem. We call it a discrete Dido-type question since it is related to the well-known Dido-problem of Hajós ([3], [4], [5]) and also it is related to the problem of [2], but we believe it to be a new question.

Definition 1. A subset S of the Euclidean plane is polygonally connected if given any two points X and Y in S there exist points $X_0 = X, X_1, \dots, X_{k-1}, X_k = Y$ such that $P = \bigcup_{i=1}^{k} \overline{X_{i-1}X_i}$ is contained in S, where $\overline{X_{i-1}X_i}$ is the segment joining X_{i-1} and $X_i (1 \leqq i \leqq k)$. The set P is called a polygonal path from X to Y.

Problem. Suppose that we have a finite number of segments in the Euclidean plane such that they form a polygonally connected subset of the plane (Fig. 1). Provided that we may not change the lengths of our segments find the polygonally connected arrangement the area of the convex hull of which is maximal.

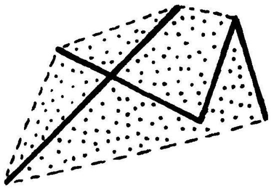

Figure 1

Conjecture. The extremal arrangement is the polygonal path of the segments which is inscribed a hemicircle (Fig. 2a).
Of course the order of the segments in this polygonal path can be arbitrary. Also, it seems to be true that the polygonal paths mentioned above are the only extremal arrangements except the case of three segments (Fig. 2b).

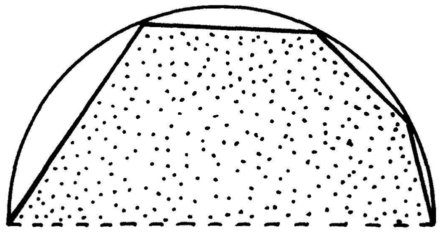

Figure 2a

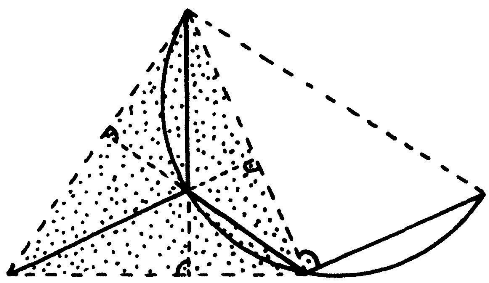

Figure 2b

In the present note we are going to prove the following two theorems, the first of which supports our conjecture and the second of which shows that our problem can lead to some interesting configurations in the higher dimensional Euclidean spaces as well.

Definition 2. A graph is simple if it does not contain loops or parallel edges, and a graph is connected if for any two vertices there exists a path of the edges from one vertex to the other.

Theorem 1. Let G_n be an arbitrary connected simple graph of n edges ($n \geqq 4$) embedded in the Euclidean plane such that the edges are segments. If GH_n is the polygonal path of n segments which is inscribed a hemicircle and the segments of which are congruent to the n segments of G_n, then the area $\mathscr{A}(\operatorname{conv} G_n)$ of the convex hull $\operatorname{conv} G_n$ of G_n is smaller than or equal to the area $\mathscr{A}(\operatorname{conv} GH_n)$ of the convex hull $\operatorname{conv} GH_n$ of GH_n with equality if and only if G_n is a polygonal path inscribed a hemicircle (Fig. 3).

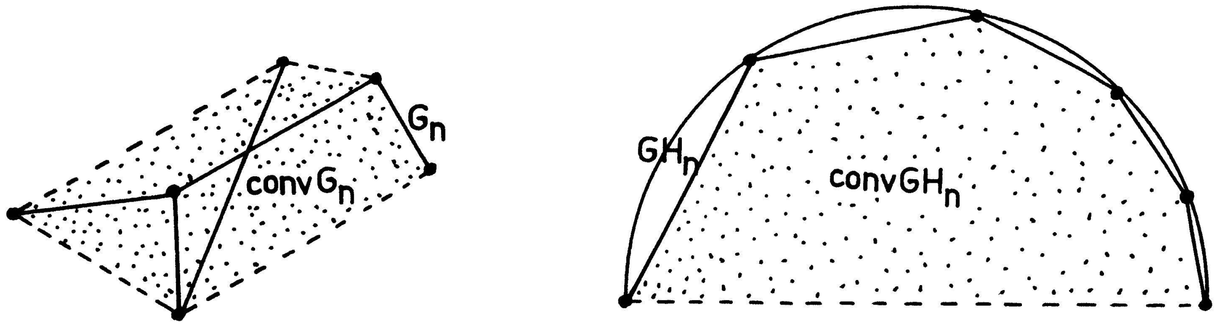

Figure 3

Remark 1. In Theorem 1 the set G_n of n segments is obviously a polygonally connected subset of the Euclidean plane. However the converse is not true i.e. there are polygonally connected arrangements of n segments in the plane which cannot be represented as G_n-sets. This shows the difference between Theorem 1 and our conjecture.

Theorem 2. Let G^d_{d+1} be an arbitrary connected simple graph of $d+1$ edges embedded in the d-dimensional Euclidean space ($d \geqq 2$) such that the edges are segments. If GS^d_{d+1} is the star formed by the $d+1$ segments of G^d_{d+1} where the center of the star GS^d_{d+1} is in the interior or conv GS^d_{d+1} and is the center of the altitudes of the simplex the vertices of which are the endpoints of GS^d_{d+1} (Fig. 4), then for the d-dimensional volumes of conv G^d_{d+1} and conv GS^d_{d+1} we have the inequality

$$V(\operatorname{conv} G^d_{d+1}) \leqq V(\operatorname{conv} GS^d_{d+1}).$$

Remark 2. It is easy to see that the inequalities $\mathscr{A}(D) \leqq \frac{1}{2 \cdot \pi} \cdot L^2(\Gamma)$, $\mathscr{A}(\operatorname{conv} \Gamma) \leqq \frac{1}{2 \cdot \pi} \cdot L^2(\Gamma)$ of the introduction are simple corollaries of Theorem 1. So also the well-known isoperimetric property of the circles follows from Theorem 1.

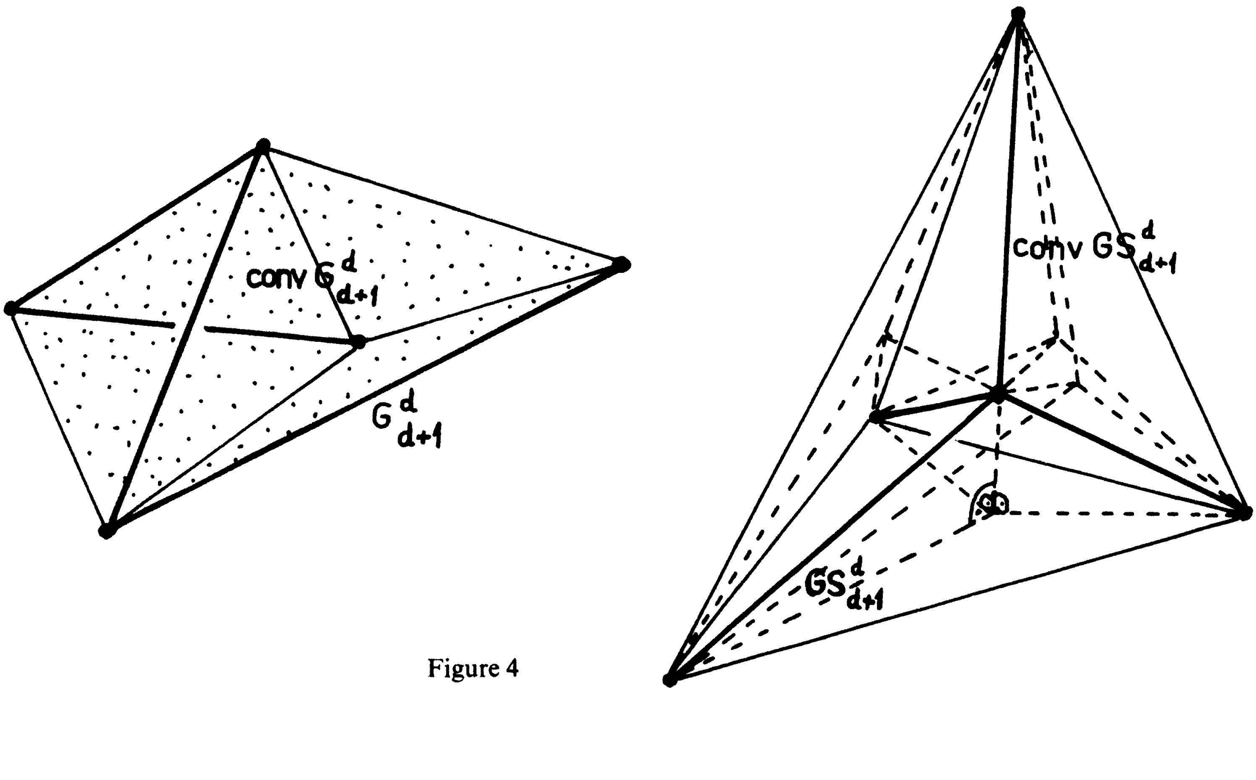

Figure 4

First let us see the proof of Theorem 1. It is an easy exercise to show that

$$\mathscr{A}(\operatorname{conv} G_3) \leqq \mathscr{A}(\operatorname{conv} GH_3). \tag{1}$$

Let $\mathscr{C}_n = \{G_n | G_n$ is a connected simple graph of n edges embedded in the Euclidean plane such that the edges are segments of the given n lengths$\}$. Because of the theorem of Weierstrass there exists a $G_n^* \in \mathscr{C}_n$ such that $\mathscr{A}(\operatorname{conv} G_n) \leqq \mathscr{A}(\operatorname{conv} G_n^*)$ for any $G_n \in \mathscr{C}_n$. We are going to show that G_n^* is a polygonal path inscribed a hemicircle.
Furtheron we suppose that $n \geqq 4$ and because of (1) we may suppose the inequality

$$\mathscr{A}(\operatorname{conv} G_{n-1}) \leqq \mathscr{A}(\operatorname{conv} GH_{n-1}) \tag{2}$$

also. From those we prove that G_n^* is a polygonal path inscribed a hemicircle, which then proves Theorem 1.

Proposition 1. G_n^* is a tree.

Proposition 2. If V is a vertex of degree one of the graph G_n^*, then V is a vertex of the convex hull of G_n^*.
The proofs of these two propositions are easy exercises which can be left to the reader.

Proposition 3. If V_1 and V_2 are two vertices of degree one of the graph G_n^*, then they are consecutive vertices (of $\operatorname{conv} G_n^*$) on the boundary of $\operatorname{conv} G_n^*$.

Proof: Suppose on the contrary that V_1, V_2 are two vertices of degree one of the graph G_n^* which are not consecutive vertices of $\operatorname{conv} G_n^*$ on the boundary of $\operatorname{conv} G_n^*$. This means that there are vertices $U_1^{(1)}, U_1^{(2)}, U_2^{(1)}, U_2^{(2)}$ of the convex hull of G_n^* such that $U_1^{(1)}, V_1, U_1^{(2)}$ is a triplet of consecutive vertices and also $U_2^{(1)}, V_2, U_2^{(2)}$ is another triplet of consecutive vertices of $\operatorname{conv} G_n^*$ (Fig. 5).

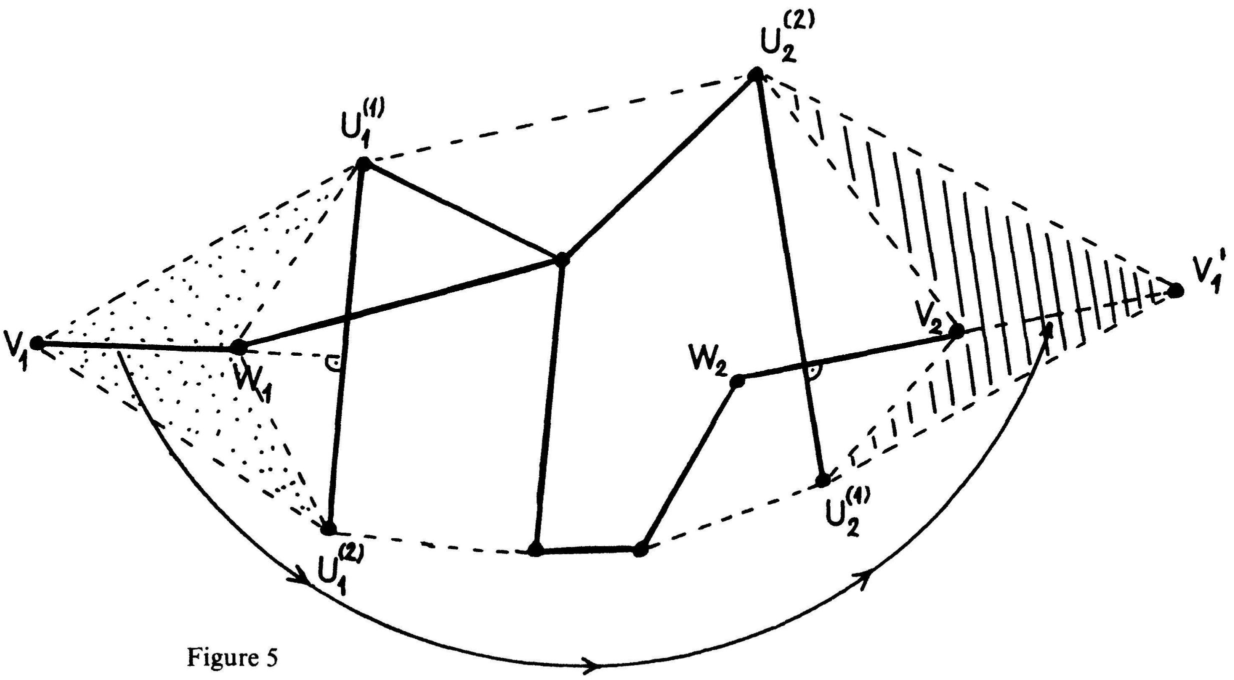

Figure 5

Obviously the edge $\overline{V_1 W_1}$ $(\overline{V_2 W_2})$ of G_n^* is orthogonal to the line $U_1^{(1)} U_1^{(2)}$ $(U_2^{(1)} U_2^{(2)})$. Without loss of generality we may suppose that the lengths of the segments $\overline{U_1^{(1)} U_1^{(2)}}$, $\overline{U_2^{(1)} U_2^{(2)}}$ satisfy the inequality $\overline{U_1^{(1)} U_1^{(2)}} \leqq \overline{U_2^{(1)} U_2^{(2)}}$. Now let V_2 be the interior point of the segment $\overline{W_2 V_1'}$ such that $\overline{V_2 V_1'} = \overline{V_1 W_1}$. In other words we put the segment $\overline{V_1 W_1}$ in a new position namely, in $\overline{V_2 V_1'}$, which obviously yields a new graph $G_n^{*\prime} \in \mathscr{C}_n$. It is easy to see that

$$\mathscr{A}(\operatorname{conv} G_n^{*\prime}) - \mathscr{A}(\operatorname{conv} G_n^*) \geqq \tfrac{1}{2} \cdot \overline{V_1 W_1} \cdot (\overline{U_2^{(1)} U_2^{(2)}} - \overline{U_1^{(1)} U_1^{(2)}}) \geqq 0\,. \tag{3}$$

But $G_n^{*\prime}$ is a connected simple graph of $(n-1)$ edges in the Euclidean plane where the edges are segments of the given $(n-1)$ lengths, since the degree of V_2 was one in G_n^*. Hence, because of (2), we have

$$\mathscr{A}(\operatorname{conv} G_n^{*\prime}) \leqq \mathscr{A}(\operatorname{conv} GH_{n-1}) \tag{4}$$

where GH_{n-1} is the polygonal path formed by the $(n-1)$ segments of $G_n^{*\prime}$, inscribed a hemicircle such that the last segment is $\overline{W_2 V_1'}$ (Fig. 6)

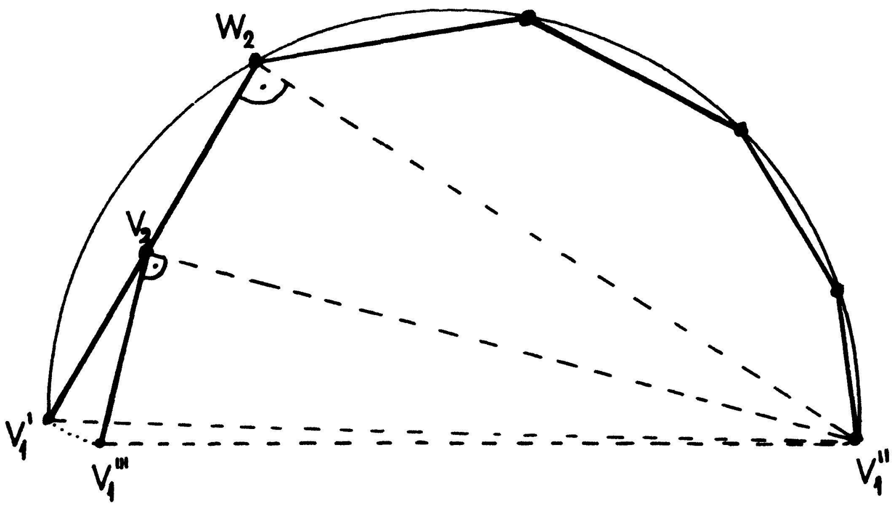

Figure 6

Let V_1'' be the other endpoint of the diameter of the hemicircle of GH_{n-1}. Here $\sphericalangle\, V_1' W_2 V_1'' = \frac{\pi}{2}$ and so $\sphericalangle\, V_1' V_2 V_1'' > \frac{\pi}{2}$ consequently we can rotate $\overline{V_2 V_1'}$ about the point V_2 into the new position $\overline{V_2 V_1'''}$ such that the arising polygonal path $G_n^{*\prime\prime} \in \mathscr{C}_n$ satisfies the inequality

$$\mathscr{A}(\operatorname{conv} GH_{n-1}) < \mathscr{A}(\operatorname{conv} G_n^{*\prime\prime})\,. \tag{5}$$

Thus on account of (3), (4), (5) we get that $\mathscr{A}(\operatorname{conv} G_n^*) < \mathscr{A}(\operatorname{conv} G_n^{*\prime\prime})$ with $G_n^*, G_n^{*\prime\prime} \in \mathscr{C}_n$ which is a contradiction. □

Proposition 4. The total number of the vertices of the graph G_n^* the degree of which is equal to one is two.

Proof: Because of the Proposition 3 the total number of the vertices of the graph G_n^* the degree of which is equal to one is at most three (and of course is at least two). Now

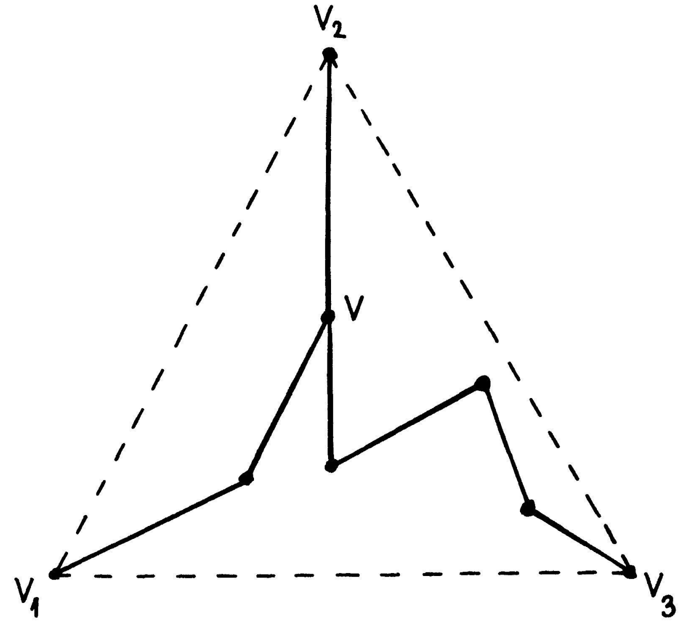

Figure 7

suppose that G_n^* possesses three vertices of degree one. On account of the Proposition 3 the convex hull of the graph G_n^* will be the triangle $\triangle V_1 V_2 V_3$ where V_1, V_2, V_3 are the vertices of degree one in G_n^* (Fig. 7).
Because of the Proposition 1 and 2 the graph G_n^* possesses one vertex V with degree three and each vertex different from V_1, V_2, V_3, V has degree two. Considering the path of the graph G_n^* from V to $V_i (i = 1, 2, 3)$ it has to be the segment $\overline{VV_i}$ otherwise we could increase the area of the convex hull of G_n^*. Also, the segment $\overline{VV_i}$ is perpendicular to the side $\overline{V_j V_k}$ of the triangle $\triangle V_1 V_2 V_3 (\{i,j,k\} = \{1,2,3\})$. Finally at least one of the segments $\overline{VV_1}$, $\overline{VV_2}$, $\overline{VV_3}$ consists of at least two edges of G_n^* because $n \geqq 4$ (Fig. 8). This clearly yields a contradiction, namely it is enough to apply the method of Fig. 6 to the configuration of Fig. 8. □

Now the rest of the proof of Theorem 1 is more or less a routine exercise. Namely,

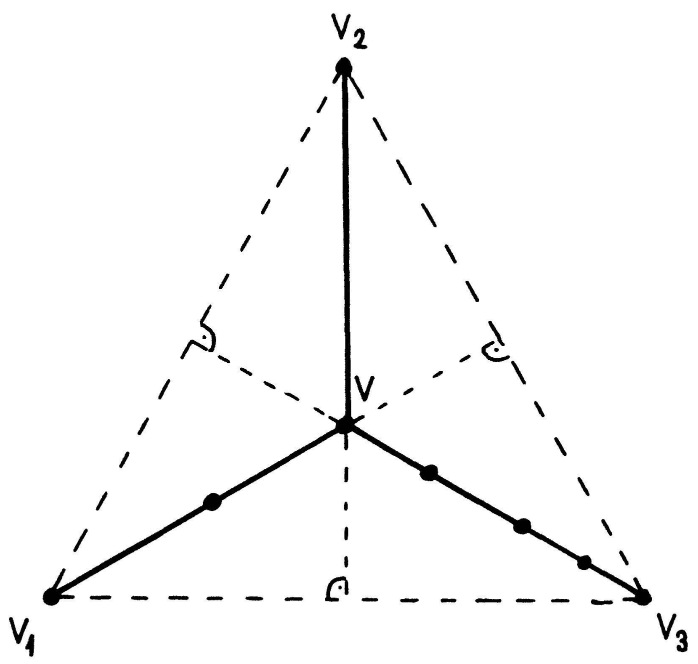

Figure 8

Proposition 5. G_n^* is a convex polygonal path of n segments.

Proof: From the Proposition 4 we get that G_n^* has two vertices V_1 and V_2 with degree one and all the other vertices have the degree two. In addition V_1 and V_2 are consecutive vertices of conv G_n^* on the boundary of conv G_n^* (Proposition 3). We claim that

$$G_n^* = b\,d(\mathrm{conv}\,G_n^*)\backslash]\,V_1, V_2[\tag{6}$$

where $b\,d(\ldots)$ means the boundary of the corresponding set and $]\ldots,\ldots[$ means the corresponding open segment. If (6) were not true, then as the Fig. 9 shows a simple reflection about a point or any other transformation which preserves the lengths of the edges of G_n^* and the connectivity of G_n^* could increase the area of the convex hull of G_n^* which would yield a contradiction. □

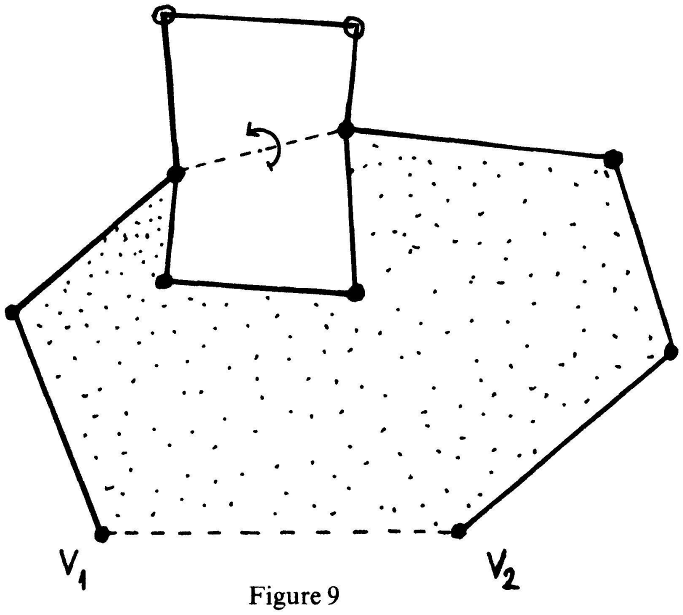

Figure 9

Proposition 6. G_n^* is a polygonal path of n segments of the given n lengths which is inscribed a hemicircle.

Proof: Using the notations of the previous proof it is enough to show that if X is an arbitrary vertex of G_n^* different from V_1, V_2, then $\sphericalangle V_1 X V_2 = \frac{\pi}{2}$. Because of the Proposition 5 the path from V_1 to X of G_n^* is a convex polygonal path and also the path from X to V_2 is a convex polygonal path. If $\sphericalangle V_1 X V_2 \neq \frac{\pi}{2}$, then a rotation about X can move the path from V_1 to X into a new position when the area of the convex hull of the new G_n^* will be larger than in the starting case which is a contradiction (see Fig. 10). □

This completes the proof of Theorem 1.
Now let us turn to the proof of Theorem 2. We sketch the main steps only without going into details.
First of all it is not hard to show that GS_{d+1}^d is uniquely determined up to congruent transformations if we know the lengths of the $d+1$ segments. On the other hand let $\mathscr{C}_{d+1}^d = \{G_{d+1}^d \mid G_{d+1}^d$ is a connected simple graph of $d+1$ edges imbedded in the d-

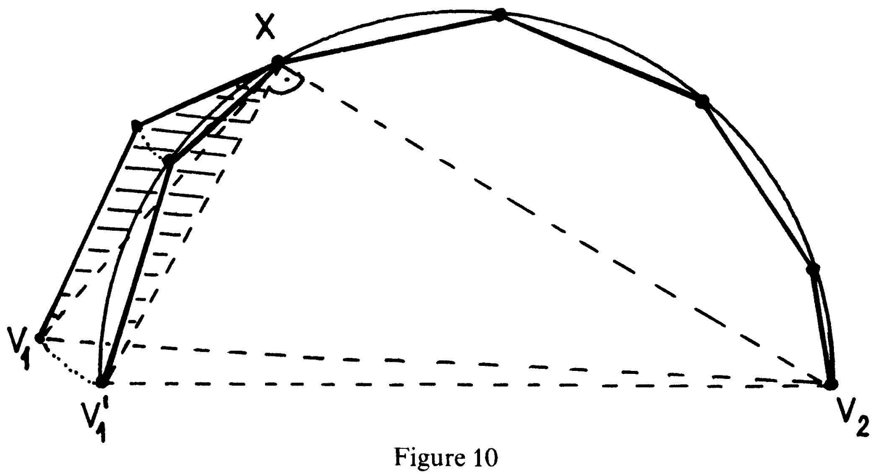

Figure 10

dimensional Euclidean space such that the edges are segments of the given $d+1$ lengths$\}$. Because of the theorem of Weierstrass there exists a $G^{*d}_{d+1} \in \mathscr{C}^d_{d+1}$ such that $V(\operatorname{conv} G^d_{d+1}) \leqq V(\operatorname{conv} G^{*d}_{d+1})$ for any $G^d_{d+1} \in \mathscr{C}^d_{d+1}$. We claim that $V(\operatorname{conv} G^{*d}_{d+1}) = V(GS^d_{d+1})$. We prove this with the help of the following transformation which transforms G^{*d}_{d+1} into a graph of $\mathscr{C}^d_{d+1}$ which is a star of $(d+1)$ segments of the given $d+1$ lengths and the volume of the convex hull of which is equal to $V(\operatorname{conv} G^{*d}_{d+1})$. From this it follows immediately that the center of the star is in the interior of the convex hull of the star and so it must be the center of the altitudes of the simplex whose vertices are the endpoints of the star. Finally because of our first observation we get that the star in question is congruent to GS^d_{d+1} and so $V(\operatorname{conv} G^{*d}_{d+1}) = V(\operatorname{conv} GS^d_{d+1})$ really, which yields Theorem 2.

The promised transformation is the composition of finite many transformations which increase the maximal degree of the graphs in question by one. Now let us see how it happens. We have a graph of $\mathscr{C}^d_{d+1}$ say G^{*d}_{d+1}, the volume of the convex hull of which is maximal in $\mathscr{C}^d_{d+1}$. Suppose that V is a vertex of the maximal degree in G^{*d}_{d+1}. We may suppose that there exists an edge $\overline{U_1 U_2}$ of G^{*d}_{d+1} whose endpoints U_1, U_2 are different from V, otherwise we are done. Also we may suppose that $G = G^{*d}_{d+1} \setminus \overline{U_1 U_2}$ is a connected simple graph of d edges imbedded in the d-dimensional Euclidean space ($d \geqq 2$) such that the edges are segments i.e. we may suppose that the degree of U_2 is one. If $\dim(\operatorname{conv} G) \leqq d-1$, then we translate the edge $\overline{U_1 U_2}$ by the vector $\overrightarrow{U_1 V}$ to the vertex V, which obviously yields a graph G^* of $\mathscr{C}^d_{d+1}$ the maximal degree of which is larger than the maximal degree of G^{*d}_{d+1} by one and finally $V(\operatorname{conv} G^*) = V(\operatorname{conv} G^{*d}_{d+1})$. If $\dim(\operatorname{conv} G) = d$, then $\operatorname{conv} G$ is a d-dimensional simplex because it is the convex hull of d (line) segments forming a connected simple graph G of d edges in the d-dimensional Euclidean space ($d \geqq 2$). Now V is a vertex of $\operatorname{conv} G$. Consider the parallel illumination of the simplex $\operatorname{conv} G$ determined by the direction $\overrightarrow{U_1 U_2}$ (Fig. 11).

Let V_f be the facet of $\operatorname{conv} G$ opposite to V. If the facet V_f is illuminated (i.e. for any interior point of V_f there exists a ray of the illumination parallel to $\overrightarrow{U_1 U_2}$ which intersects V_f at the given interior point going into the interior of $\operatorname{conv} G$), then we translate the edge $\overline{U_1 U_2}$ of the graph G^{*d}_{d+1} by the vector $\overrightarrow{U_1 V}$ to the vertex V otherwise we translate $\overline{U_1 U_2}$ by the vector $\overrightarrow{U_2 V}$ to the vertex V. Let $\overline{VV^*}$ be the new edge (segment) at the vertex V in both cases forming a new graph G^* of $\mathscr{C}^d_{d+1}$ together with G. Finally let us denote the orthogonal projection of $\operatorname{conv} G$ onto the hyperplane H by $P(\operatorname{conv} G)$ where H is a hyperplane orthogonal to the line $U_1 U_2$. It is not hard to show that

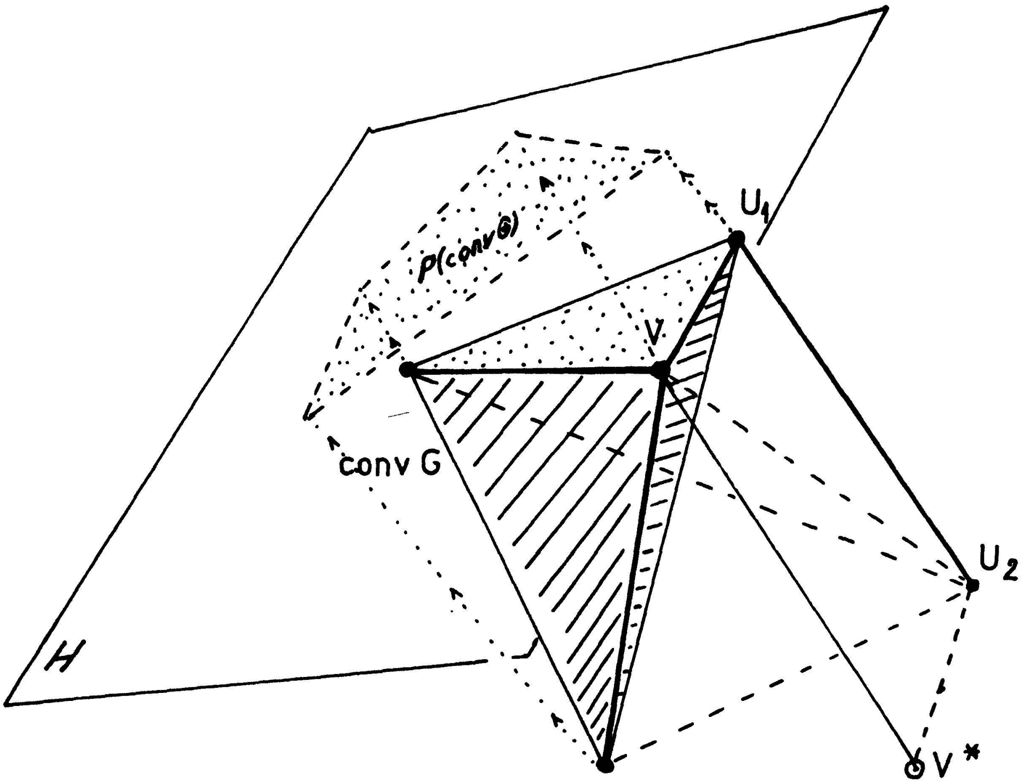

Figure 11

$$V(\operatorname{conv} G^*) = V(\operatorname{conv} G) + \frac{1}{d} \cdot \mathscr{A}[P(\operatorname{conv} G)] \cdot \overline{VV^*}$$

$$= V(\operatorname{conv} G) + \frac{1}{d} \cdot \mathscr{A}[P(\operatorname{conv} G)] \cdot \overline{U_1 U_2}$$

where $\mathscr{A}(\ldots)$ means the $(d-1)$-dimensional volume of the corresponding set) and $V(\operatorname{conv} G^{*d}_{d+1}) \leqq V(\operatorname{conv} G) + \frac{1}{d} \cdot \mathscr{A}[P(\operatorname{conv} G)] \cdot \overline{U_1 U_2}$. Hence $V(\operatorname{conv} G^*) = V(\operatorname{conv} G^{*d}_{d+1})$ where G^* is a graph of $\mathscr{C}^d_{d+1}$ the maximal degree of which is larger than the maximal degree of G^{*d}_{d+1} by one.
This completes the proof of Theorem 2.

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