# A characterization of the tangent function

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Objekttyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **44 (1989)**

Heft 4

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-41616

# A characterization of the tangent function

Real-valued functions $T$ satisfying the identity $T(u+v)=\dfrac{T(u)+T(v)}{1-T(u)T(v)}$ are studied. It is proved that the tangent function is the only such function having domain $\{x{:}x \text{ real}, x \neq \frac{\pi}{2}+m\pi, m \text{ an integer}\}$ and satisfying $T'(0)=1$.

During the past decade, several of our articles ([1], [2], [3], [4], [5]) have suggested a more fundamental role for the tangent function, tan, in the curriculum. Many of these suggestions depend on the fact that $\tan(u+v)=\dfrac{\tan u+\tan v}{1-\tan u\tan v}$ whenever the right-hand side is defined. It seems natural to ask whether this functional identity characterizes tan. Accordingly, this note considers the class of *tangential functions*, by which we mean real-valued functions $T$ of a real variable such that $T(u+v)=\dfrac{T(u)+T(v)}{1-T(u)T(v)}$ whenever the right-hand side is defined. In proposition 1(c), we produce infinitely many discontinuous tangential functions, thus answering the above question in the negative. On the other hand, Theorem 3 establishes that tan is the only tangential function $T$ which is defined at all real numbers other than $\frac{\pi}{2}+m\pi$ (for $m$ an integer) and satisfies $T'(0)=1$. We hope that the material in this note will find use as enrichment material in introductory courses on calculus.
We begin by collecting some examples of tangential functions.

**Proposition 1.** Each of the following functions $T$ is tangential:

(a) $T(x)=1$ for each real number $x$;
(b) $T(x)=-1$ for each real number $x$;
(c) Let $p$ be a fixed prime number. If $x$ is a real number, put

$$T(x)=\begin{cases}0 & \text{if} \quad x=\dfrac{m}{p^n} \text{ for some integers } m \text{ and } n\\ 1 & \text{otherwise}.\end{cases}$$

*Proof.* (a) and (b): The functional identity holds by default since $1-T(u)T(v)\equiv 0$ means that the identity's right-hand side is never defined.
(c): We shall verify the functional identity. Without loss of generality, $T(u)T(v)\neq 1$. Thus at least one of $u$, $v$ is of the form $\dfrac{m}{p^n}$. If both $u$ and $v$ have this form, so does $u+v$, in which case the functional identity reduces to the truism $0=\dfrac{0+0}{1-0}$. On the other hand, if only one of $u$, $v$ has the form $\dfrac{m}{p^n}$, then $u+v$ does not have this form, in which case the identity reduces to either $1=\dfrac{1+0}{1-0}$ or $1=\dfrac{0+1}{1-0}$. ■

A tangential function need not be continuous (and, hence, need not be differentiable).

Indeed, if $T$ is as in Proposition 1(c), then for each real number $c$, $\lim_{x \to c} T(x)$ does not exist. In Proposition 2(b), (c), we examine what can be said about a tangential function which is continuous (or differentiable).

**Proposition 2.** Let $T$ be a tangential function. Then:

(a) If $T(0)$ is either 1 or $-1$, then $T(x) = T(0)$ for each $x$ in the domain of $T$.

(b) Suppose that the domain of $T$ contains a neighbourhood of 0. If $T$ is continuous at 0, then $T$ is a continuous function.

(c) Suppose that the domain of $T$ contains a neighbourhood of 0 and that $T'(0) = 1$. Then $T$ is a differentiable function. In fact, $T'(x) = 1 + T(x)^2$ for each $x$ in the domain of $T$. Moreover, $T(0) = 0$ and $T$ is increasing on each subinterval of its domain.

*Proof.* (a) Suppose $T(0) = \pm 1$ and $T(x) \neq T(0)$. Then $1 - T(x)T(0) \neq 0$, and so the functional identity of tangential functions leads to

$$T(x) = T(x+0) = \frac{T(x) + T(0)}{1 - T(x)T(0)} = \frac{T(x) \pm 1}{1 \mp T(x)},$$

whence $T(x)[1 \mp T(x)] = T(x) \pm 1$ and $T(x)^2 = -1$. This contradicts the fact that $T$ is real-valued

(b) Since constant functions are continuous, (a) allows us to suppose that $T(0) \neq \pm 1$. Hence $1 - T(0)^2 \neq 0$, and so the functional identity gives

$$T(0) = T(0+0) = \frac{T(0) + T(0)}{1 - T(0)^2},$$

whence $T(0)[1 - T(0)^2] = 2\,T(0)$ and $0 = T(0)^3 + T(0) = T(0)[T(0)^2 + 1]$. As $T(0)^2 + 1 \neq 0$, we have $T(0) = 0$. By hypothesis, $\lim_{h \to 0} T(h) = T(0) = 0$. For each $x$ in the domain of $T$,

$$\lim_{h \to 0} T(x+h) - T(x) = \lim_{h \to 0} \frac{T(x) + T(h)}{1 - T(x)T(h)} - T(x) = \lim_{h \to 0} \frac{[1 + T(x)^2]\,T(h)}{1 - T(x)T(h)},$$

which, by limit theorems, is just $\dfrac{[1 + T(x)^2]0}{1 - (T(x))0} = 0$. Thus $\lim_{h \to 0} T(x+h) = T(x)$, and so $T$ is continuous at $x$, proving (b).

(c) Since $T'(0) \neq 0$, it follows from (a) that $T(0) \neq \pm 1$. Hence, by the proof of (b), we have $T(0) = 0$. It follows that $\lim_{h \to 0} \dfrac{T(h)}{h} = \lim_{h \to 0} \dfrac{T(h) - T(0)}{h} = T'(0) = 1$. Now, for each $x$ in the domain of $T$, we see, as in the proof of (b), that

$$\begin{aligned} \lim_{h \to 0} \frac{T(x+h) - T(x)}{h} &= \lim_{h \to 0} [1 + T(x)^2] \frac{T(h)}{h} \left[\frac{1}{1 - T(x)T(h)}\right] \\ &= [1 + T(x)^2] \cdot 1 \cdot \left[\frac{1}{1 - T(x) \cdot 0}\right]; \end{aligned}$$

that is, $T'(x) = 1 + T(x)^2$. In particular, $T'(x) > 0$. The final assertion is a standard consequence of the Mean Value Theorem. ■

We next obtain the desired characterization of tan. For motivation, note that $\tan'(0) = \sec^2(0) = 1^2 = 1$.

**Theorem 3.** *Let $T$ be a tangential function such that $T'(0) = 1$ and each real number $x \neq \frac{\pi}{2} + m\pi$ (for $m$ an integer) is in the domain of $T$. Then $T = \tan$.*

*Proof.* First, we restrict attention to $x$ in the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. By Proposition 2(c), $T$ satisfies the variables-separable differential equation $y' = 1 + y^2$. This leads to

$$x = x - 0 = \int_0^x dt = \int_{T(0)}^{T(x)} \frac{ds}{1+s^2} = \int_0^{T(x)} \frac{ds}{1+s^2}$$
$$= \tan^{-1}(T(x)) - \tan^{-1}(0) = \tan^{-1}(T(x)) - 0 = \tan^{-1}(T(x)).$$

Hence, $T(x) = \tan(\tan^{-1}(T(x))) = \tan(x)$ for all $x$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$. [Remark: A short classroom discussion might well end here, as we have just used/reinforced the fundamental theorem of calculus and the chain rule, in the guise of definite integration by change of variable.]
Next, we focus on $\frac{\pi}{2} < x (\neq \frac{\pi}{2} + m\pi)$. First, suppose $\frac{\pi}{2} < x < \pi$. Then $x = 2u = u + v$, where $u = v$ is in $(\frac{\pi}{4}, \frac{\pi}{2})$. As $T$ and tan are both tangential, we may argue as follows, using the result of the preceding paragraph:

$$T(x) = T(u+v) = \frac{T(u) + T(v)}{1 - T(u)\,T(v)} = \frac{\tan(u) + \tan(v)}{1 - \tan(u)\tan(v)} = \tan(u+v) = \tan(x).$$

Moreover, Proposition 2(b) yields that $T$ is continuous, and so, since tan is also continuous, we have $T(\pi) = \lim_{x \to \pi^-} T(x) = \lim_{x \to \pi^-} \tan(x) = \tan(\pi) = 0$. Hence, by mathematical induction (and the fact that $T$ is tangential), we have $T(n\pi) = 0$ for each positive integer $n$. It now follows, by reasoning as above, that $T$ and tan agree on $(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2})$. Indeed, if $x$ is in this interval, then $x - n\pi = \tan^{-1}(T(x)) - \tan^{-1}(T(n\pi))$, whence $T(x) = \tan(x - n\pi) = \tan(x)$. Hence, $T$ and tan agree on $(-\frac{\pi}{2}, \infty)\backslash\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\}$.
Next, we focus on $-\frac{\pi}{2} > x (\neq \frac{\pi}{2} + m\pi)$. First, suppose $-\pi < x < -\frac{\pi}{2}$. Then $x = 2u = u + v$, where $u = v$ is in $(-\frac{\pi}{2}, -\frac{\pi}{4})$. In particular, $T(u) = \tan(u)$. It follows via tangentiality as above that $T(x) = T(u+v) = \tan(u+v) = \tan(x)$. Then, by considering the limit as $x$ approaches $-\pi$ from the right and invoking continuity, we see that $T(-\pi) = 0$. By mathematical induction and tangentiality, $T(-n\pi) = 0$ for each positive integer $n$. It now follows, by reasoning as above, that $T$ and tan agree on $(-n\pi - \frac{\pi}{2}, -n\pi + \frac{\pi}{2})$. Hence, $T$ and tan agree on the domain of tan.
We have seen that $T(x) = \tan(x)$ if $x \neq \frac{\pi}{2} + m\pi$ (for $m$ an integer). To complete the proof, it suffices to show that $T(\frac{\pi}{2} + m\pi)$ is not defined. This, however, is a consequence of the continuity of $T$, since $\lim_{x \to \frac{\pi}{2} + m\pi} T(x) = \lim_{x \to \frac{\pi}{2} + m\pi} \tan(x)$ does not exist. ■

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 0013-6018/89/040101-04$1.50 + 0.20/0

# Kleine Mitteilungen

## An explicit formula about the convex hull of random points

Denote by $V_n^{(d)}$ the expected volume of the convex hull of $n$ points chosen independently according to a given probability measure $\mu$ in Euclidean $d$-space $E^d$. For $d = 2, 3$ and $\mu$ the uniform distribution on a convex body in $E^d$, Affentranger [1], [2] has shown that

$$V_{d+2m}^{(d)} = \sum_{k=1}^{m} \gamma_k \binom{d+2m}{2k-1} V_{d+2m-2k+1}^{(d)} \quad (m = 1, 2, \ldots), \tag{1}$$

where the $\gamma_k$ can be obtained recursively from $\gamma_1 = \frac{1}{2}$, $2\gamma_k = 1 - \sum_{i=1}^{k-1} \binom{2k-1}{2i-1} \gamma_i \, (k \geq 2)$.

Recently, Buchta [3] has extended this result to arbitrary dimensions $d$ and to arbitrary probability measures $\mu$ on $E^d$. The key point in [3] is the existence of a moment functional $\mathcal{M}$ such that

$$V_{d+1+n}^{(d)} = \binom{d+1+n}{d+1} \mathcal{M}(x^n + (1-x)^n). \tag{2}$$

(See [4] for the definition of moment functionals.)
In this note we show that in formula (1) the $\gamma_k$ can be expressed explicitly by

$$\gamma_k = (2^{2k} - 1) \frac{B_{2k}}{k} \quad (k = 1, 2, \ldots). \tag{3}$$

Here the $B_n$ are the *Bernoulli numbers* (see e.g. [5], section 1.13), defined by the generating series $z/(e^z - 1) = \sum_{n=0}^{\infty} B_n z^n/n!$. In our proof of formula (1) we can avoid the elimination process used in [1].