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## LITERATURVERZEICHNIS

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## Kleine Mitteilung

## Zeros of characters and the Frattini subgroup

Let  $G$  be a finite group and let  $\text{Irr}(G)$  be the set of its (complex) irreducible characters. Of course, the Frattini subgroup  $\Phi(G)$  being normal in  $G$ , it must be an intersection of certain kernels of elements of  $\text{Irr}(G)$ . However, it seems that the problem of describing  $\Phi(G)$  in terms of characters is still open. The aim of this short Note is to give a sufficient condition for the nontriviality of  $\Phi(G)$  in terms of vanishing sets of nonlinear irreducible characters of  $G$ .

Our notation is standard and follows that of [2]. Throughout,  $G$  will be a finite group and  $Z(G)$ ,  $G'$  will denote its centre and its derived subgroup, respectively. If  $\chi \in \text{Irr}(G)$  is nonlinear, the vanishing set  $A(\chi)$  of  $\chi$  is defined by  $A(\chi) := \{g \in G / \chi(g) = 0\}$ . A well-known result of Burnside asserts that  $A(\chi) \neq \Phi$ ; moreover, it's clear that  $A(\chi)$  is a union of conjugate classes of elements of  $G$ .

We prove the following

**Theorem.** *Let  $G$  be a finite group with  $1 < Z(G) < G$ . Suppose that there exists a nonlinear  $\chi \in \text{Irr}(G)$  such that  $A(\chi)$  contains fewer than  $|Z(G)|$  conjugate classes of elements. Then  $\Phi(G) \neq 1$ .*

**Proof:** The key observation is that actually  $A(\chi)$  is a union of cosets modulo  $Z(G)$ . To prove this, note that by Problem 3.12 of [2] it follows that for every  $g \in G$ ,

$$|\chi(g)|^2 = \frac{\chi(1)}{|G|} \sum_{h \in G} \chi([g, h]). \quad (*)$$

Let now  $g, h \in G$  and  $z \in Z(G)$ ; since  $[g, h] = [gz, h]$ , it follows from (\*) that  $g \in A(\chi)$  iff  $gz \in A(\chi)$  for every  $z \in Z(G)$ . This means that  $A(\chi)$  is a union of cosets modulo  $Z(G)$ .

Suppose, by way of contradiction, that  $\Phi(G) = 1$ . By a well-known result of [1],  $G' \cap Z(G) \leq \Phi(G)$ , so  $G' \cap Z(G) = 1$ .

Denote by  $s(\chi)$  and  $t(\chi)$  the number of conjugate classes of  $G$  contained in  $A(\chi)$  and the number of cosets modulo  $Z(G)$  lying in  $A(\chi)$ , respectively. We shall reach a contradiction by applying the pigeonhole principle. Suppose that  $g, h \in A(\chi)$ ,  $g \neq h$  and there exist  $u \in G$  and  $z \in Z(G)$  such that  $g = h^u = hz$ . Then  $z = h^{-1}g = h^{-1}h^u = [h, u] \in G' \cap Z(G)$ , whence  $g = h$ . This contradicts the choice of  $g$  and  $h$  and shows that  $A(\chi)$  contains at most  $s(\chi)t(\chi)$

elements. Taking now into account that  $t(\chi) := |A(\chi)|/|Z(G)|$ , it results that  $|Z(G)| \geq s(\chi)$ . But this contradicts the hypothesis and we are done.

**Corollary.** *Let  $G$  be a finite group with  $1 < Z(G) < G$  and  $\Phi(G) = 1$ . Let*

$$c := \max \{ |C_G(x)| / |x \in G \setminus Z(G)| \} \quad \text{and} \quad b(G) := \max \{ \chi(1) / \chi \in \text{Irr}(G) \}.$$

Then 
$$b(G) \geq \left( \frac{c}{c - |Z(G)|} \right)^{\frac{1}{2}}.$$

**Proof:** As a direct consequence of the Theorem, we obtain that  $c|A(\chi)| \geq |Z(G)||G|$  for every nonlinear  $\chi \in \text{Irr}(G)$ . On the other hand, it's a simple exercise to prove that  $|A(\chi)| < |G| \frac{\chi(1)^2 - 1}{\chi(1)^2}$ . Since  $\frac{(b(G))^2 - 1}{(b(G))^2} \geq \frac{\chi(1)^2 - 1}{\chi(1)^2}$  for every  $\chi \in \text{Irr}(G)$ , the result follows by combining these inequalities.

We have already seen that if  $\chi \in \text{Irr}(G)$  is nonlinear, then  $|Z(G)| \mid |A(\chi)|$ . This result may be refined in certain very special cases. For example, suppose that  $G$  is a finite group and  $\chi \in \text{Irr}(G)$  is faithful such that  $\chi(1) = 2$  and  $\chi(g) \in \mathbb{Q}$  for every  $g \in G$ . It is a matter of simple calculations to show that  $|A(\chi)| = 3|Z(G)|$ . If, moreover,  $Z(G) = 1$ , then  $|A(\chi)| = 3$  and  $G$  has a maximal subgroup of index 3 (the centralizer of an element of  $A(\chi)$ ).

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## Aufgaben

**Aufgabe 1009.**  $n$  Zahlen  $x_1, \dots, x_n$  mit  $x_i \in \{0, 1, \dots, k\}$  ( $k \geq 2$ ) werden einmal linear, ein andermal kreisförmig so angeordnet, dass die Summe zweier Nachbarglieder stets von  $k + 1$  verschieden ist. Für beide Fälle bestimme man die Anzahl der zulässigen Anordnungen.

J. Binz, Bolligen

**Lösung des Aufgabenstellers** (Bearbeitung der Redaktion).

$s_n$  bzw.  $t_n$  bezeichnen die Anzahlen der zulässigen linearen bzw. kreisförmigen Anordnungen.

a) Es gilt  $s_n = a_n + b_n$ , wobei  $a_n$  die Anzahl der auf 0 endenden,  $b_n$  diejenige der übrigen zulässigen linearen Anordnungen bedeuten. Dann ist

$$a_{n+1} = a_n + b_n, \quad b_{n+1} = k a_n + (k - 1) b_n, \quad a_2 = k + 1, \quad b_2 = k^2,$$