§ 2. The Connection between Lie Groups and Reflection Groups.

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In stating the theorems in § 3, we give explicitly the restrictions that must be placed on the ground field k and also on the group W. For each theorem it is briefly indicated how the result may be proved. Sometimes a proof in general terms is known, but in the majority of cases it has been necessary to verify the properties one by one for all the irreducible groups over C and then show that (with the exception of (vi)) they extend to the reducible groups. These two distinct methods will be referred to as *proving* and *verifying* respectively. Perhaps the most remarkable fact is that the result (iv) of § 3 which appears to concern itself entirely with discrete infinite groups generated by reflections has been proved only by topological methods (spectral sequences and Morse theory). A direct proof, avoiding the topology, would be interesting. Further outstanding problems are the discovery of proofs for those properties that have so far only been verified, and the extension of these theorems to more general fields k, especially to the case where k is of finite characteristic.

§ 2. The Connection between Lie Groups and Reflection Groups.

Let G be an n dimensional compact semi-simple Lie Group. A maximal connected abelian subgroup of G forms a submanifold of G which is a torus of dimension r (the *rank* of G) [10]. This is called a *maximal torus* T of G. The inner automorphisms of G by elements of N_T, the normaliser of T, induce a finite group of automorphisms of T. These in turn induce linear transformations of the tangent space V^r to T at the identity e. It can be shown that this group of linear transformations forms a reflection group over R called the *Weyl group* W of G. This group has the further property that it is *crystallographic*, i.e. by suitable choice of coordinates it is represented by a set of matrices whose coefficients are integers, or, alternatively, if the coordinates are chosen so that the matrices are orthogonal (so that W is then a group of congruent transformations acting on a Euclidean space R^r) then the angle between any two hyperplanes of reflection of W is an integral multiple of $\pi/4$ or $\pi/6$.

The converse is also true, namely that any crystallographic reflection group over R corresponds to some compact semi-simple Lie Group.

It can be shown that T may be covered by a Euclidean space R^r in such a way that the *singular elements* of T (i.e. those whose normalisers are of dimension strictly greater than r) map into hyperplanes of R^r, and further, if the identity of G maps into the origin O of R^r, then those planes passing through O are precisely the hyperplanes of reflection of the Weyl group W. The whole set of hyperplanes form a configuration known as the *diagram* of the Lie Group G and it has the property that reflection in any one of the planes leaves the diagram, as a whole invariant.

Now let W be *any* reflection group over R expressed in orthogonal form, then W may be considered as operating on some sphere S^{r-1} whose centre is at O. The hyperplanes of reflection divide the surface of the sphere into spherical polytopes and it has been shown [5; p. 190] that each of these is necessarily a simplex or a direct product of simplexes. Further, the r hyperplanes that cut the sphere in the faces of one of these polytopes form a *fundamental set* in that the corresponding reflections generate the group. Furthermore, the volume bounded by these hyperplanes forms a fundamental region for W. A property of this fundamental set is given in (vi) of § 3.

Considering again the diagram of the Lie Group G, pick out a fundamental set of hyperplanes through O, defining a fundamental region of the Weyl group. Then the part of the diagram of G that lies within this fundamental region is called a *Weyl chamber*. The Weyl chamber of the group G_2 (the group of automorphisms of the Cayley matrix algebra) is illustrated in (iv) of § 3, where, for the present, the numerals are to be ignored.

§ 3. Properties of the Exponents.

(i) The ring of polynomial invariants of an r dimensional reflection group W over k is the ring $k[I_1, I_2, \dots, I_r]$ where I_i is a polynomial invariant of degree $(m_i + 1)$. The I_i are uniquely determined by this property and are called the *basic invariants* of the group W. The m_i are called the *exponents* of W.