THE POSSIBLE TRANSFORMATIONS OF A REAL CURVE INTO A CURVE WITH REAL EQUATION AND PASSING THROUGH THE ISOTROPIC POINTS

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THE POSSIBLE TRANSFORMATIONS OF A REAL CURVE INTO A CURVE WITH REAL EQUATION AND PASSING THROUGH THE ISOTROPIC POINTS

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1. The first problem of this paper is to investigate the possibility of transforming a curve, expressible as a function with real coefficients of the variables, into a second curve expressible by an equation in cartesian coordinates with real coefficients and such that two real points on the curve become the isotropic points. It should be noted that any real transformation of the coordinate system will not affect the reality of the equation of a curve.

Let A and B be two real points on such a curve, and suppose $f(x, y, z) = 0$ is the equation of the curve referred to a real triangle of reference XYZ where X, Y are harmonic conjugates with respect to A, B. The lines ZA, ZB are then given by $y \pm \lambda x = 0$, where λ is some real constant. It is easy to show that if the equations of ZA, ZB are to become $\iota y \pm x = 0$, the necessary transform is equivalent to that in which x is replaced by x, y by $\lambda \iota y$ and z by ιz, where the triangle of reference remains unchanged. X and real points on YZ are the only points which remain real, and the only real lines which remain real are YZ and those through X. If we then replace z by unity, ZA and ZB become isotropic lines, $\iota y + x = 0$, in a cartesian field with real rectangular axes ZX, ZY. A and B are now the isotropic points in this field.

Suppose that in the original field we have a non-degenerate curve of degree N with real coefficients and of the form

$$(x, y, z) \equiv (U_0 x^N + U_2 x^{N-2} + U_4 x^{N-4} + \ldots) + (U_1 x^{N-1} + \\ + U_3 x^{N-3} + U_5 x^{N-5} + \ldots) = 0 \, .$$

where U_r is homogeneous of degree r in y and z. After the first transform the curve has equation

$$(u_0 x^{N} - u_2 x^{N-2} + u_4 x^{N-4} - \dots) + \iota (u_1 x^{N-1} - u_3 x^{N-3} + \\ + u_5 x^{N-5} - \dots) = 0 ,$$

where u_r is homogeneous of degree r in y and z and with real coefficients. Hence, if the curve is to transform finally into one through the isotropic points and whose equation has real coefficients, the equation must initially have the form when $N = 2n$

$$U_0 x^{2n} + U_2 x^{2n-2} + \dots\dots + U_{2n-2} x^2 + U_{2n} = 0 , \qquad (1)$$

and when $N = 2n + 1$

$$U_1 x^{2n} + U_3 x^{2n-2} + \dots\dots + U_{2n-1} x^2 + U_{2n+1} = 0 . \qquad (2)$$

In the first case all the odd polar curves with respect to X have x as a factor, with YZ as the polar line of X. In the second case the curve passes through X, which is an inflexion: the odd polar curves with respect to X have x as a factor (corresponding to the line YZ), the remaining part of the $(2n - 1)$th. polar being the inflexional tangent $U_1 = 0$. We will refer to YZ as the conjugate line of X in each case. If the curve has a multiple point of order k at X, its equation if of the form

$$U_k x^{N-k} + U_{k+2} x^{N-k-2} + \dots + U_N = 0 ,$$

with $N - k$ even. The $(k + 1)$th. polar curve of X is $U_k x = 0$, that is the tangents at the multiple point (of necessity inflexional) and the conjugate line.

Rewriting the equations in descending powers of z, we have

$$V_0 z^{2n} + V_1 z^{2n-1} + \dots + V_r z^{2n-r} + \dots + V_{2n} = 0 , \qquad (3)$$

or

$$V_0 z^{2n+1} + V_1 z^{2n} + \dots + V_r z^{2n-r+1} + \dots + V_{2n+1} = 0 , \qquad (4)$$

where V_r is homogeneous of degree r in x and y. Since only even powers of x occur it follows that V_r is of the form

$$\prod_{1}^{r/2} (a_s x^2 + b_s y^2) \quad \text{when } r \text{ is even,}$$

or

$$y \prod_{1}^{r/2} (a_s x^2 + b_s y^2) \quad \text{when } r \text{ is odd.}$$

Hence the set of points on XY, other than X, given by any $V_r = 0$ form an involution with X, Y as double points. In particular the points of intersection of the curve with XY, given by $V_N = 0$, have the same property. The expression of $y^2 - \lambda^2 x^2$ on which the transformation was based will be a factor of V_N, that is one at least of the expressions a_s/b_s is negative, and A, B are any pair of points corresponding to such a factor $a_s x^2 + b_s y^2$. It follows that any curve with real equation which satisfies these conditions can be transformed into a circular curve with real equation.

2. Consider the intersections of any curve of this type with the line $py + qz = 0$, joining X to any point P on YZ. Their joins to Z are given by an equation of the same form as $V_N = 0$, and hence the intersections, other than X, form an involution with X and P as the double points. It follows at once that if P is a $2k + 1$ ple point on the curve XP is a tangent to one of the branches. We shall refer to any real point having the properties of X with respect to the curve as a pole. We have now shown that, given a pole, any real point on the curve and its mate in the involution cut on the line joining the point to the pole can be transformed into the isotropic points, giving a circular curve whose equation has real coefficients.

If V_r occurs in the equation of the curve, the equation of the $(N - r)$th. polar curve with respect to Z is of the same form as (3), (4): if V_r does not occur the equation of the $(N - r)$th. polar curve with respect to Z contains z to some power as a factor and the remaining part is of the same form as (3, 4). X is therefore a pole of the $(N - r)$th. polar curve of Z or of the remaining part of it, as the case may be, where Z is any point on the conjugate line of X. Hence since the first polar curve of Z passes once through each node and not, in general through a point of contact of a tangent from X, a node and such a point of contact cannot be paired together, and similarly for other singularities. The case of two nodes collinear with X in which one of them only has XY as a tangent similarly cannot arise. The two points which are to become the isotropic points must therefore be of exactly similar type. Further any isolated singularity must be at X or on YZ.

It should be noticed that in the case of a curve of even degree containing only even powers of z a pole at X implies also poles at Y and Z.

3. The second problem is to discuss the possibility of a second pole not on YZ. From the preceding results it follows that the form of the equation of the curve having a pole at X is unaltered if the triangle of reference is defined by X and any two real points on YZ. If the curve has a second pole X_1 not on YZ, we can therefore take it as lying on XY with Z the common point of the conjugate lines of X and X_1. The intersection of these lines cannot be at Y since in that case X, Y and X_1, Y would be double points of two involutions among the same points (projecting Y to infinity, X and X_1 would both become the mean centre of the same points). The conjugate lines of X and X_1 are therefore distinct and lead to a definite point of intersection Z.

It follows at once that the points X and X_1 are poles for all polar curves with respect to Z of the initial curve and lie on all such curves of odd degree. The points of intersection of $V_r = 0$ with XY for any value of r (but excluding X if r is odd) may therefore be paired as an involution having X as a double point (the other double point being Y) or, alternatively excluding X_1 if r is odd they may be paired as an involution having X_1 as a double point. In this case the second double point is the intersection, Y_1, of the conjugate line of X_1 with XY. Further, since a real point cannot be paired with an imaginary point in any involution with real double points, nor can a point of simple intersection be paired with one of multiple intersection, the various types of points must be grouped together to form sub-groups. Each sub-group must separately form involutions with X, Y and X_1, Y_1 as double points. The expression leading to a sub-group of s points will be denoted by v_s having the same form as V_s and

$$V_r \equiv v_s \,.\, v_t \,\ldots \quad \text{where} \quad r = s + t + \ldots$$

in which some of $s, t, \ldots$ may be equal. The existence of a second pair of double points X_1, Y_1 entails further restriction on the form of φ_r.

Project Y to infinity and pairs of points on XY given by $\varphi_r = 0$ become symmetric with respect to X.

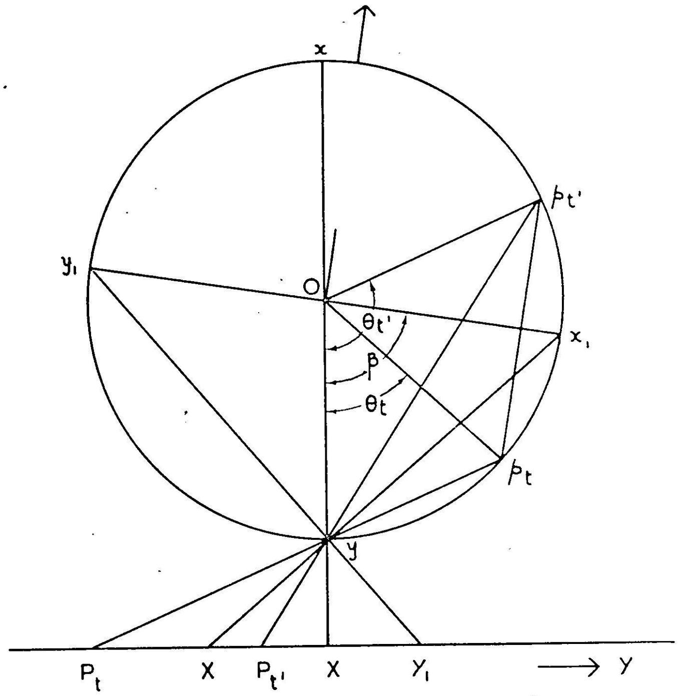

FIGURE I.

Taking $r = 2\rho + 1$, X_1 is a point of φ_r and since there must be ρ points between X_1 and Y_1, X must separate X_1 and Y_1. Taking $r = 2\rho$ $(\rho > 1)$, X divides the points into two equal groups (α) and (β) and X_1, Y_1 are the double points of an involution among $(\alpha) + (\beta)$ in some order. If X_1 and Y_1 are not separated by X, let them be on the side of the group (α). Since there must be ρ points of φ_r between X_1 and Y_1, they must be the points (α). The harmonic conjugates with respect to X_1, Y_1 of an ordered set of points from X_1 to Y_1 are external to $X_1 Y_1$ and in the reverse order, that is they are the set (β). Any point of (α) is therefore paired with the same point of (β) in both involutions and therefore the premise is incorrect and X must separate X_1 and Y_1. Let one of the intersections of the perpendicular to XY at X with the circle on $X_1 Y_1$ as diameter be y. Since X is between X_1 and Y_1 this point is real. Draw any circle through y with its centre, O, on the normal Xy

produced through y, and meeting the normal again in x. Denote any point of the involution by P_t where $2l - 1$, $2l$ are the suffixes of a pair of the involution with double points X and Y. In the odd degree case X is itself a point of $\varphi_r = 0$ and may then be denoted by P_0. Projecting from y as vertex on to the circle points P_t will give points p_t which form corresponding involutions on the circle. The centre of the involution on the circle corresponding to the involution with X, Y as double points is Y at infinity. X_1, Y_1 become points x_1, y_1 at the ends of a diameter of the circle, and therefore the centre of this involution is at infinity in the direction perpendicular to $x_1 y_1$, and the joins of pairs of points of the second involution on the circle must be parallel to this direction. Let the angle between the directions of the two centres of involution which equals yOx_1 be β. Let yp_t subtend the angle θ_t at the centre of the circle.

(i) Even degree, φ_{2r} $(r > 1)$.

From the involution on the circle formed by the parallel chords through Y

$$\begin{aligned} \theta_1 + \theta_2 &= 0 \pmod{2\pi} \\ \theta_3 + \theta_4 &= 0 \pmod{2\pi} \\ &\cdots\cdots \\ &\cdots\cdots \\ \theta_{2r-1} + \theta_{2r} &= 0 \pmod{2\pi} , \end{aligned} \tag{5}$$

and adding

$$\sum_1^{2r} \theta_t = 0 \pmod{2\pi} .$$

Pairing for the involution defined by the double points X_1 and Y_1 we have similarly, from parallel chords, r equations

$$\theta_t + \theta_{t'} = 2\beta \pmod{2\pi} \tag{6}$$

in which t, t' take the values 1, 2, 3, ... $2r$ $(t \neq t')$. Hence

$$\beta = \frac{2s\pi}{2r} = \frac{s\pi}{r} ,$$

where s can take any one of the values 1 to $r - 1$.

If p_t is the mate of p_l in the first involution and of p_m in the second, we have

$$\theta_l + \theta_t = 0 \pmod{2\pi}, \quad \theta_m + \theta_t = 2\beta \pmod{2\pi},$$
$$\therefore \theta_m - \theta_l = 2\beta \pmod{2\pi},$$

and the angle between yp_m and yp_l is the constant β of the second involution. Since β can take $r - 1$ values, it follows that if there are two involutions among the $2r$ points, there are r involutions.

The argument used to prove that X separates X_1, Y_1 also demonstrates that between any two successive points of φ_{2r} there cannot be more than one double point (X or Y) of the possible hyperbolic involutions. Hence the r involutions now determined form the complete set of hyperbolic involutions among the points.

(ii) Odd degree, φ_{2r+1}.

It has already been shown that in the odd case any pole is itself a point of the group and is a double point of an involution among the remaining points. Without any loss of generality we can therefore take X at P_0 and any possible second pole X_1 at P_1. Then for the first involution we have $\theta_0 = \pi \pmod{2\pi}$ in addition to equations (5). For the second involution we have $\theta_1 = \beta \pmod{2\pi}$ in addition to equations (6) in which t, t' take the values 0, 2, 3 ... $2r$, $(t \neq t')$. This finally leads to

$$\beta = \frac{2s + 1}{2r + 1} \pi ;$$

where s can take any of the values 0 to $2r$. Hence if two involutions can be formed in this manner, $2r + 1$ involutions can be so formed. Also the angle $P_0 yP$, for any P, is a value of $(\pi - \beta)/2$.

There are $2r + 1$ positions for the pole X and Y is fixed uniquely as the harmonic conjugate of X with respect to the two points of φ_{2r+1} nearest to X.

4. We now determine the basic algebraic forms for φ_r. Suppose P is any intersection of φ_r with XY of the triangle XYZ, given by a factor $mx - y = 0$ of φ_r. Then when YZ is projected to great distance P on XY is given by $m - y = 0$. Hence the corresponding factor of φ_r in the XYZ field is $x \sin\theta - y \cos\theta$, where $\theta = \sphericalangle XyP$, apart from a possible scale adjustment.

(i) It has been shown that for all polar curves, $\varphi_r = 0$, of degree > 2 with more than one pole the centres of the involutions on the circle are at great distance. A curve may also have a polar curve of degree 2, $\varphi_2 = 0$, and hence for φ_2 we only consider possible centres of involution at great distance. Since the join of the two points on the circle has to pass through two points at great distance, the two points themselves must be at great distance and are therefore the isotropic points and

$$\varphi_2 \propto x^2 + y^2 .$$

Any other point on the line at great distance is a possible centre.

(ii) φ_{2r} $(r > 1)$ with the corresponding points on XY all real.

Denote the angle XyP, where P is any of the points, by δ; then

$$\varphi_{2r} \propto \prod_{0}^{r-1} \left[x^2 \sin^2\left(\delta + \frac{s\pi}{r}\right) - y^2 \cos^2\left(\delta + \frac{s\pi}{r}\right)\right],$$

$$\propto (x^2 + y^2)^r \prod_{0}^{r-1} \left[\cos\left(2\delta + \frac{2s\pi}{r}\right) - \cos 2\Phi\right] \text{ where } \tan\Phi = y/x ,$$

$$\propto (x^2 + y^2)^r [\cos 2r\delta - \cos 2r\Phi] ,$$

$$\propto (x^2 + y^2)^r \cos 2r\delta - x^{2r}\left[1 - \frac{2r(2r-1)}{2!}\tan^2\Phi + \frac{2r(2r-1)(2r-2)(2r-3)}{4!}\tan^4\Phi \ldots\right],$$

$$\propto (x^2 + y^2)^r \cos 2r\delta - \left[x^{2r} - \frac{2r(2r-1)}{2!}x^{2r-2}y^2 + \frac{2r(2r-1)(2r-2)(2r-3)}{4!}x^{2r-4}y^4 - \ldots\right].$$

(iii) φ_{2r} $(r > 1)$ with the corresponding points on XY all imaginary and distinct.

The product of $x \sin\left(\theta + \frac{s\pi}{r}\right) + y\cos\left(\theta + \frac{s\pi}{r}\right)$, where $\theta = p + \iota q$ (p and q being real and $q \neq 0$), and its conjugate gives the expression

$$(x^2 + y^2)\left[\cosh 2q - \cos\left(\Phi + p + \frac{s\pi}{r}\right)\right] \Big/ 2$$

where $\tan\Phi = y/x$. Giving s the values 0, 1, ... $(r - 1)$ the product of the resulting expressions is proportional to

$$(x^2 + y^2)^r [\cosh 2rq - \cos 2r(\Phi + p)]$$
$$= (x^2 + y^2)^r \cosh 2rq - \cos 2rp \left[x^{2r} - \frac{2r(2r-1)}{2!} x^{2r-2} y^2 + \dots\right]$$
$$+ y \sin 2rp \left[2rx^{2r-1} - \frac{2r(2r-1)(2r-2)}{3!} x^{2r-3} y^2 + \dots\right].$$

Now φ_{2r} contains even powers of x only, and hence we must take $p = t\pi/2r$, where t is an integer, and $\cos 2rp = \pm 1$.

Giving s its run of values and $t = 0$, we get from $x \sin\left(\theta + \frac{s\pi}{r}\right) + y \cos\left(\theta + \frac{s\pi}{r}\right)$ one set of points and a second set from their conjugates. Changing t by an even number interchanges each set within itself: changing t by an odd number interchanges the sets completely. Hence it is sufficient to take the two values 0 and 1 of t, obtaining for φ_{2r}

$$(x^2 + y^2)^r \cosh 2rq - \left[x^{2r} - \frac{2r(2r-1)}{2!} x^{2r-2} y^2 + \dots\right] (t = 0),$$

or

$$(x^2 + y^2)^r \cosh 2rq + \left[x^{2r} - \frac{2r(2r-1)}{2!} x^{2r-2} y^2 + \dots\right] (t = 1).$$

(The mate of a point s in one set is given by $s' = 2r - 4t - s$ in the other.)

An inclusive form for φ_{2r} is

$$a(x^2 + y^2)^r + b\left[x^{2r} - \frac{2r(2r-1)}{2!} x^{2r-2} y^2 + \dots\right].$$

(iv) φ_{2r+1} with the corresponding points on XY all real and distinct.

$$\varphi_{2r+1} \propto y \prod_0^{r-1} \left[x^2 \sin^2 \frac{(r-s)\pi}{2r+1} - y^2 \cos^2 \frac{(r-s)\pi}{2r+1}\right],$$

$$\propto y x^{2r} \prod_0^{r-1} \left[\tan^2 \frac{(r-s)\pi}{2r+1} + \left(\frac{\iota y}{x}\right)^2\right],$$

$$\propto \frac{(x + \iota y)^{2r+1} - (x - \iota y)^{2r+1}}{2\iota},$$

and

$$\varphi_{2r+1} \propto y \left[x^{2r} - \frac{2r(2r-1)}{3!} x^{2r-2} y^2 + \dots\right].$$

We have so far considered only the basic forms of φ. We now consider the possible combinations and special forms of φ which may occur in a complete V.

5. If some of the intersections of V_r are real and some are imaginary (all being supposed distinct), then V_r must be replaced by $V_s\, V_{r-s}$ where V_s is built up from basic φ's with real intersections and V_{r-s} from basic φ's with imaginary intersections.

The points of a φ_{2r} will not be distinct if for some value or values of

$$\delta,\ s,\ s'\ (s' \neq s)$$

$$\tan\,(\delta + s\pi/r) = \pm \tan\,(\delta + s'\pi/r)\ ,$$

that is

$$\delta + s\pi/r = \pm\,(\delta + s'\pi/r)\ (\mathrm{mod}\ \pi)\ ,$$

or

$$2r\delta = 0\ (\mathrm{mod}\ \pi)\ .$$

This implies that the points are all repeated and are at r alternate poles of φ_{2r}. These points may be given by a $(\varphi_r)^2$ with $\cos\, r\delta=0$, $\cos\, 2r\delta = -1$ in the case when r is even or with $\cos\, 2r\delta = +1$ when r is odd, and must then be replaced by φ_{2r}.

Again φ_2 to any power may occur in V and for V_r we must have $\varphi_2^l\, V_{r-2l}$.

The modifications required for any combination or extension of these circumstances or for peculiarities of higher orders are manifest. It can readily be shown that

$$\varphi_{2p(\delta)}\,\varphi_{2p\left(\delta+\frac{\pi}{pq}\right)} \cdots \varphi_{2p\left(\delta+\frac{(q-1)\pi}{pq}\right)} = \varphi_{2pq(\delta)}\ ,$$

where the portion of the suffix in brackets denotes the value of " δ " for the attached φ_{2p} and defines the position of the points of the involution. The corresponding factor of V is then φ_{2pq}.

It can also be shown that

$$\varphi_{(2p+1)}\,\varphi_{2(2p+1)\{\pi/(2p+1)(2r+1)\}} \cdots \varphi_{2(2p+1)\{r\pi/(2p+1)(2r+1)\}} = \varphi_{(2p+1)(2r+1)}\,.$$

If p, q are odd and p is prime the expressions φ_{pq} and φ_{2mp} cannot have a common factor unless the p poles of φ_p are points of φ_{2mp}

V_r and φ_r will be referred to as of order r and are homogeneous of degree r in x and y.

6. We now consider the assemblage of terms which form $f\,(x, y, z)$, $\Sigma\alpha_r\, V_r\, z^{N-r}$, where the α's are constants, and the V's are of the forms previously specified. Possible involutions determined by each V_r may be displayed on a circle, and the only possible poles for the complete curve are given by the common involution centres of all φ's which occur. The isotropic points given by $\varphi_2 = 0$ belong to every involution. The number of possible poles is therefore equal to the highest common factor, p, of the orders of the φ's, not including φ_2 and its powers. This implies that for any p the form of V_N imposes conditions on the existence or the form of any V_{N-k}. If p is even the number of involutions is $p/2$ and both double points of each involution are possible poles, and the conjugate line of any such point meets XY in a possible pole. If p is odd the number of involutions is p, and only one double point of each involution is a possible pole: the intersection of the conjugate line of such a point with XY is not a pole. A possible pole will actually be a pole if its conjugate line passes through Z.

The form of the function can be slightly modified by a real scale change.

7. Consider a curve having a pole at X and with a multiplicity of order $k \geqslant 0$ at X. From the basic forms (1) and (2) if the curve is of even degree, k must be even and if the curve is of odd degree, k must be odd. In both cases $N - k$ is therefore even. We will also suppose that there are further possible poles on XY. Since X is a k-ple point no powers of x greater than $N - k$ may occur in the equation of the curve and this power must occur.

V_{N-k+t}, t 0 to k, if it occurs, must contain y to the power t at least as a factor and " at least " must be replaced by exactly for one or more values of t. A V which contains y^t as a factor is necessarily the product of a φ's of odd order $2m_s + 1$, s 1 to a, and b φ's of order $2(2n_s + 1)$, s 1 to b, $\cos 2(2n + 1)\delta = 1$, where $a + 2b = t$. The total number of poles (actual and possible) on XY is a factor of $2m_s + 1$ and $2(2n_s + 1)$ and hence this number may be taken as p or $2p$ where p is odd. From paragraph 5 it follows that φ_p is a single factor of each of the a φ's of odd order and a repeated factor of each of the b φ's of even order, and the corresponding V is the product of $(\varphi_p)^t$ and an expression ϖ. We can now write the equation of the curve in the form

$$f(x, y, z) \equiv \alpha_0 z^N + \alpha_2 z^{N-2} v_2 + \dots + \alpha_{N-k-1} z^{k+1} V_{N-k-1}$$
$$+ \sum_{t=0}^{t=k} \alpha_{N-k+t} z^{k-t} (v_p)^{t+\tau} w_t = 0 ,$$

where τ is zero or a positive integer for each term. A term containing x^{N-k} is obtained when τ is zero and must occur at least once.

If P is any possible pole on XY and P' its conjugate phase point, then the harmonic conjugate of X with respect to P, P' is also a possible pole and a k-ple point. Hence when the number of possible poles on XY is odd every such point is a k-ple point and when the number is even every alternate point is a k-ple point. In each case these possible poles (at k-ple points) are the intersections of XY with $v_p = 0$. Let X' $(x' : y' : 0)$ be such a point. The tangents at X' are given by

$$\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right)^k f(x, y, z) = 0 ,$$

and these together with the conjugate line of X' are given by

$$\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right)^{k+1} f(x, y, z) = 0 ,$$

where x, y, z, are replaced by x', y', o after differentiation and X, Y, Z, are here used to represent the current coordinates. Now

$$\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right)^k z^{k-t} (v_p)^{t+\tau} w_t$$
$$= \frac{k\,!}{t\,!\,(k - t)\,!} Z^{k-t} (k - t)\,! \left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}\right)^t (v_p)^{t+\tau} w_t ,$$

on putting $z = 0$ after differentiation. The expression

$$\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}\right)^t (v_p)^{t+\tau} w_t$$

vanishes except when $\tau = 0$ since $v_p = 0$ when $x = x'$, $y = y'$. The tangent form is obtained from those values of t for which the corresponding τ's are zero. If t' is the least of these values, the highest power of Z is $k - t'$ and the tangents consist of $k - t'$ lines not through Z and $X'Z$ counted t' times.

Now V_{N-k-1} is of odd order, hence if the number of possible poles is odd the term $\alpha_{N-k-1}\, z^{k+1}\, V_{N-k-1}$ either contains ϱ_p as a factor or $\alpha_{N-k-1} = 0$, and if the number is even α_{N-k-1} must be zero. In either case the term does not appear after the substitution of x_1, y_1 for x, y and therefore makes no contribution to the tangent and conjugate line form. We now obtain

$$\frac{(k+1)!}{(t+1)!\,(k-t)!} Z^{k-t} (k-t)! \left(X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y}\right)^{t+1} (\varrho_p)^{t+\tau} w_t \,,$$

leading in exactly the same way as before to $k - t'$ lines not through Z and $X'Z$ counted t′ times and a line also through Z. This last line must be the conjugate line of X'. It follows that the conjugate lines of all the possible poles on XY pass through Z, a point determined previously by the conjugate lines of X and a specific pole X_1. Hence all the possible poles are actual poles.

Special forms of the V's can lead to multiple points on XY which are not poles.

8. The general problem of the total number of poles or of their distribution has not been solved. From the degeneracy of the first polar curve with respect to any pole P into a curve of degree $N - 2$ and the conjugate line, the Steinerian must have a multiplicity of order $N - 2$, or equivalent singularity, at P. There is therefore, in general, an immediate crude finite limit to the number of poles. When N is odd another limitation is provided by the number of real inflexions. It has been possible however to obtain some detailed information concerning properties and numbers for certain types of curve.

We show first that any curve of even degree whose equation is symmetric in x^2, y^2, z^2 has nine poles on the sides of the triangle of reference. It's equation can be written as

$$\sum a_{rst}\, s_1^r\, s_2^s\, s_3^t = 0 \,,$$

where

$$s_1 = x^2 + y^2 + z^2 \,, \quad s_2 = y^2 z^2 + z^2 x^2 + x^2 y^2 \,, \quad s_3 = x^2 y^2 z^2 \,.$$

It clearly has poles at X, Y, Z.

Making the transformation

$$x' = x \ , \quad y' = y + z \ , \quad z' = y - z \ ,$$

leads to an equation of even degree in x', y', z', and the curve therefore has poles at X' (which is X), Y' and Z', the vertices of the new triangle of reference. Referred to the original triangle the curve has poles at $(0: 1: 1)$ and $(0: 1: -1)$. Similarly there are poles at $(1: 0: 1)$, $(-1: 0: 1)$, $(-1: 1: 0)$ and $(1: 1: 0)$.

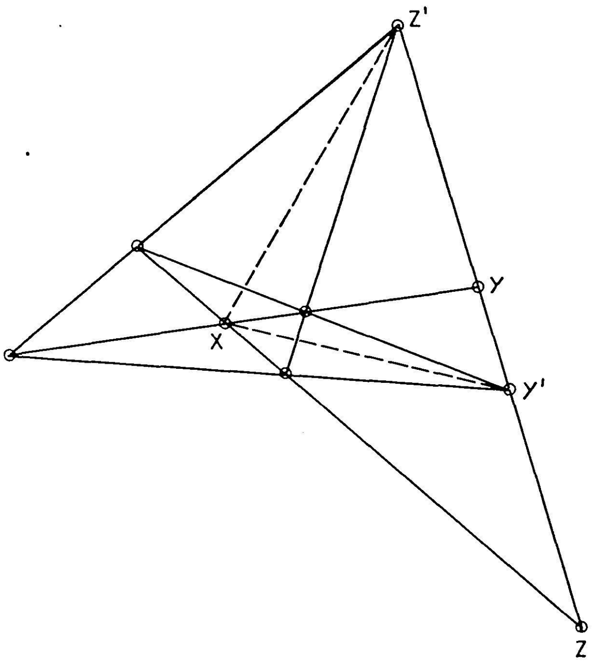

FIGURE 2.

These six new poles are collinear in threes and the lines through them form a quadrilateral of which XYZ is the harmonic triangle. Any triangle of type $XY'Z'$ is such that each side is the polar line of the opposite vertex and referred to such a triangle the curve has poles at its vertices and at points not on the sides.

We have established that if there are r poles on a straight line they can be transposed on to a circle to form an equispaced system of points. If r is even and p is the transpose of a pole P, the point p' diametrically opposite to p is also the transpose of a pole P'. If r is odd, the point p' does not correspond to a pole but to a point P' which is the double point corresponding to P of an involution formed by the intersections of the curve with the given line. In either case the conjugate line of P passes through P' and that of P' through P. Any such point,

P or P', will be referred to as a phase point, and if two phase points are such that the conjugate line of each passes through the other they will be termed conjugate phase points. From consideration of the circle it follows that if Q is a phase point, not necessarily a pole, then its harmonic conjugate with respect to P, P' is also a phase point such that both points are poles or neither are poles. It is obvious for the circle and therefore for the line that when r is odd the poles alternate with the remaining phase points.

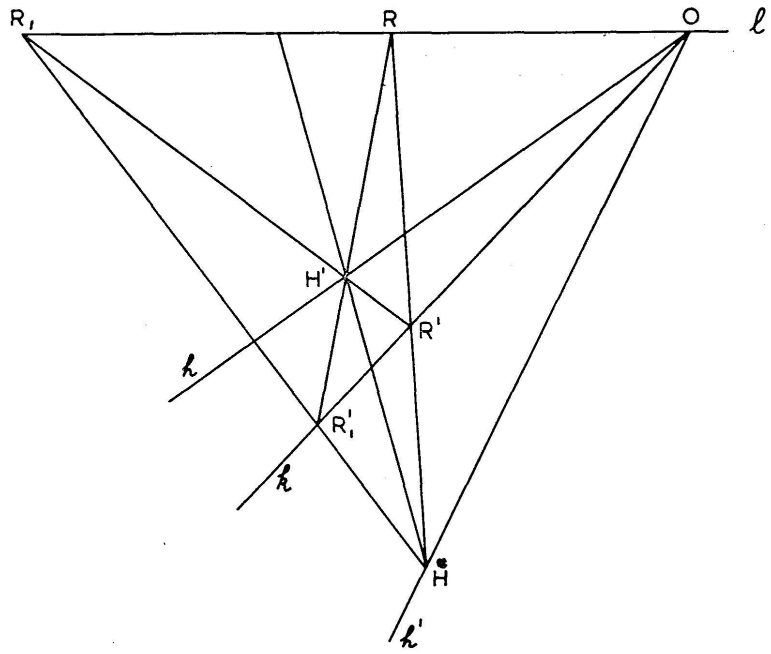

FIGURE 3.

Suppose there are r poles, R, on a line l and that H is a pole not on the line (see fig. 3). Let h be the conjugate line of H meeting l in 0 and let k be the harmonic conjugate of l with respect to OH and h. Then from the previous theorem the join of H to any pole R meets k in a point R' which is also a pole, and hence there are exactly r poles on k since the process is reversible. If O is a pole, its conjugate line will pass through H and hence meets h in a pole H'. The conjugate line of H' is OH and its k line coincides with the k line of H. HH' will pass through a phase point on l which will be a pole only if r is even.

Suppose $P_1 P_2 P_3 P_4 P_5$ is a regular pentagon with circumcentre Z and let the regular pentagon formed by the joins of alternate vertices be $Q_1 Q_2 Q_3 Q_4 Q_5$ with $P_r ZQ_r$ collinear. Let the point at great distance on the side opposite P_r be X_r, and the point at great distance on $P_r Q_r$ be Y_r. Suppose a curve can be drawn symmetric about the five lines ZP_r and with a pole at P_1 with corresponding conjugate line $Q_1 X_1$. By symmetry all the P_r's are poles. The k-line of $P_2 Q_5 Q_1 P_4 X_3$ with respect to the pole P_1 is $X_4 P_3 Q_1 Q_2 P_5$, the joins of corresponding points passing through P_1. Hence X_4, Q_5, Q_2, X_3 are poles and by symmetry all Q_r's and X_r's are poles, giving a closed system of poles. Since $P_1 Q_1 X_1$ is a self polar triangle with all the vertices poles, the curve is of even degree. The configuration has poles not on the sides of a basic triangle but no algebraic curve has been found to satisfy the primary conditions.

The joins of any pair of conjugate phase points on XY to y (fig. 1) are at right angles and therefore after projecting XY to great distance from y, the pencil of $2q$ lines joining Z to the phase points on XY forms an orthogonal involution. Calling the phase points taken in ordered sequence Z_0, Z_1, Z_2, ... Z_{2q-1}, the lines ZZ_r, ZZ_{q+r} are at right angles. Since the conjugate of Z_r with respect to Z_0 and Z_q is Z_{2q-r}, the lines ZZ_r, ZZ_{2q-r} are equally inclined to ZZ_0 and ZZ_q. All suffixes are mod $2q$. The $2q$ lines therefore form an equi-spaced system, the angle between any two successive lines being $\pi/2q$. Let L be any point on the curve and let Z_{q+r} be a pole. If L' is the image of L in ZZ_r, the conjugate line of Z_{q+r}, then ZL, ZL' are harmonically separated by ZZ_{q+r}, ZZ_r and L' also lies on the curve. The curve is therefore now symmetrical about the conjugate lines of all the poles at great distance. If H is any other phase point, necessarily finite, its kaleidoscopic images in these conjugate lines are, from the symmetry, also phase points and of the same type.