ON THE TEACHING OF GEOMETRY IN SECONDARY SCHOOLS

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16. *Multiplication of scalars and vectors. Similarity.*

16. 1. Let us consider a triangle ABC and the correspond-addition formula

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} .$$

If each of these vectors is multiplied by a scalar k one gets again a similar formula

$$k\overrightarrow{AB} + k\overrightarrow{BC} = k\overrightarrow{AC} .$$

If we replace here $\overrightarrow{AC}$ by $\overrightarrow{AB} + \overrightarrow{BC}$ one gets

$$k\overrightarrow{AB} + k\overrightarrow{BC} = k\,(\overrightarrow{AB} + \overrightarrow{BC}) .$$

In other words, the multiplication of a scalar with vectors is distributive with respect to the addition of vectors:

$$k\,(\vec{b} + \vec{c}) = k\vec{b} + k\vec{c} .$$

16. 2. If we look on the corresponding figure we see that we are dealing with 2 triangles ABC, $AB'\,C'$ and a mapping $T \rightarrow T'$ such that the ratio of the corresponding intervals be the constant k:

$$X'\,Y' : XY = k . \qquad (1)$$

As a matter of fact the mapping

$$\begin{aligned} A &\longleftrightarrow A = A' \\ B &\longleftrightarrow B' \\ C &\longleftrightarrow C' \end{aligned}$$

may be extended: for each point $\in \Delta ABC$ there is associated a single point $\in \triangle\, A'\,B'\,C'$ such that the relation (1) holds: it is sufficient to define X' in such a way that

$$\overrightarrow{AX'} = k \,.\, \overrightarrow{AX} .$$

16. 3. A similarity may be direct or indirect, according as the corresponding figures have the same or opposite orientation.

16. 4. For triangles one has 4 simple criteria (cases) of similarity:

One has the mapping $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$; moreover:

FIRST CASE: The three corresponding angles are equal: $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$;

SECOND CASE: $a' = ka$, $b' = kb$;

THIRD CASE: $a' = ka$, $b' = kb$, $a \geqslant b$, $\alpha = \alpha'$;

FOURTH CASE: $a' = ka$, $b' = kb$, $c' = kc$.

In each of these cases there is a unique extension of the above mapping such that one gets a similarity. For instance, if $X \in AB$, then X' is determined so that $X' \in A'B'$ and that $\overrightarrow{A'X'} : \overrightarrow{X'B'} = \overrightarrow{AX} : \overrightarrow{XB}$; e.g. if X is the midpoint of AB, then X' is so for $A'B'$. The corresponding angles are equal.

Distributivity of scalar multiplications of vectors.

16. 5. THEOREM. Scalar multiplication of vectors is distributive with respect to addition of vectors:

$$\vec{v}\,(\vec{a} + \vec{b}) = \vec{v}\vec{a} + \vec{v}\vec{b}\,. \tag{1}$$

PROOF. First of all, let us project $\vec{a}$, $\vec{b}$, on the vector $\vec{v}$; this projecting is distributive:

$$\text{proj}\,(\vec{a} + \vec{b}) = \text{proj}\,\vec{a} + \text{proj}\,\vec{b}\,.$$

Let us multiply this equation by the number $|\vec{v}|$:

$$|\vec{v}|\ \text{proj}\,(\vec{a} + \vec{b}) = |\vec{v}|\ (\text{proj}\,\vec{a} + \text{proj}\,\vec{b})\,.$$

At the right we have to multiply ordinary numbers; therefore one gets:

$$|\vec{v}|\ \text{pr}\,(\vec{a} + \vec{b}) = |\vec{v}|\ \text{proj}\,\vec{a} + |\vec{v}|\ \text{proj}\,\vec{b}\,.$$

Now, each of these products is the corresponding scalar product:

$$|\vec{v}|\ (\vec{a} + \vec{b}),\ \vec{v}\vec{a},\ \vec{v}\vec{b}$$

and the preceding equality yields the required formula (1).

16. 6. *General distributive law:* sum times sum is the sum of partial products each by each. E.g.

$$(\vec{a} + \vec{b})(\vec{c} + \vec{d}) = (\vec{a} + \vec{b})\vec{c} + (\vec{a} + \vec{b})\vec{d} = \vec{a}\vec{c} + \vec{b}\vec{c} + \vec{a}\vec{d} + \vec{b}\vec{d} .$$

16. 7. *Application* 1. *Pythagoras' theorem and the cosine theorem.*

For any triangle ABC we have

$$\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB} \text{ i.e.}$$

$$\vec{a} + \vec{b} = \vec{c}$$

and hence

$$(\vec{a} + \vec{b})^2 = c^2$$

i.e.

$$a^2 + b^2 + 2\vec{a}\vec{b} = c^2$$

i.e.

$$a^2 + b^2 + 2ab \cos \gamma = c^2 \qquad \textit{(cosine theorem)}$$

and for $\gamma = \pi/2$:

$$a^2 + b^2 = c^2 \qquad \textit{(Pythagoras' theorem)}.$$

16. 7. 2. *If a vector* $\vec{\mathrm{v}}$ *is orthogonal to 2 independent vectors* $\vec{\mathrm{a}}, \vec{\mathrm{b}}$ *of a plane, then it is orthogonal to every vector of this plane* p.

As a matter of fact, if $\vec{a}, \vec{b}$ are 2 independent vectors of a plane p, then for each vector $\vec{c}$ of this plane one has the unique decomposition $\vec{c} = x\vec{a} + y\vec{b}$, where x, y are numbers. Multiplying this equality by $\vec{v}$ one gets:

$$\vec{v}\vec{c} = \vec{v}(x\vec{a} + y\vec{b}) = x(\vec{v}\vec{a}) + y(\vec{v}\vec{b}) .$$

If $\vec{v}$ is orthogonal to $\vec{a}$ and $\vec{b}$ i.e. $\vec{v}\vec{a} = 0$, $\vec{v}\vec{b} = 0$, then the preceding equation yields:

$$\vec{v}\vec{c} = xO + yO = 0 \quad \text{i.e.} \quad \vec{v} \perp \vec{c} . \qquad \text{Q.E.D.}$$

16. 7. 3. COROLLARY: *If a straight line is orthogonal to 2 intersecting lines, then this line is orthogonal to the plane of these two lines.*

17. *Fundamental formula for scalar product.*

Let us consider an orthonormal system of vectors $\vec{e}_0, \vec{e}_1, \vec{e}_2$; then every vector $\vec{v}$ has a unique representation in the form

$$\vec{v} = v_0 \vec{e}_0 + v_1 \vec{e}_1 + v_2 \vec{e}_2$$

where v_0, v_1, v_2 are numbers.

Analogously, for another vector $\vec{u}$ one has the " scalar components " u_0, u_1, u_2 and the decomposition

$$\vec{u} = u_0 \vec{e}_0 + u_1 \vec{e}_1 + u_2 \vec{e}_2 .$$

The scalar product $\vec{u}\vec{v}$ becomes:

$$\vec{u}\vec{v} = (u_0 \vec{e}_0 + u_1 \vec{e}_1 + u_2 \vec{e}_2)(v_0 \vec{e}_0 + v_1 \vec{e}_1 + v_2 \vec{e}_2) = \sum_{i,k=0}^{2} u_i \vec{e}_i \cdot v_k \vec{e}_k =$$

$$= \sum_{i=0}^{2} u_i v_i \text{, because } \vec{e}_i \vec{e}_k = \begin{matrix} 0 \\ 1 \end{matrix} \text{ for } \begin{matrix} i \neq k \\ i = k \end{matrix}$$

Consequently, one has the fundamental equality:

$$|\vec{u}|\,|\vec{v}| \cos(\vec{u}, \vec{v}) = u_0 v_0 + u_1 v_1 + u_2 v_2 . \qquad (S)$$

If one is dealing with vectors in the plane $\vec{e}_0, \vec{e}_1$, then one has

$$|\vec{u}|\,|\vec{v}| \cos(\vec{u}, \vec{v}) = u_0 v_0 + u_1 v_1 . \qquad (S_2)$$

The fundamental formula (S) has important implications.
E.g. 1. The theorem of Pythagoras: let $\vec{u} = \vec{v}$; the equality (S_2) yields:

$$|\vec{u}|\,|\vec{u}| \cos \sphericalangle (\vec{u}, \vec{u}) = u_0^2 + u_1^2$$

i.e.

$$u^2 = u_0^2 + u_1^2$$

and in space

$$u^2 = u_0^2 + u_1^2 + u_3^2 , :$$

the square of the diagonal of a rectangular quader is the sum of the squares of the fundamental edges of the quader.

2. *Additional theorem for cosine.*

On the circle line let 2 points be represented by the numbers α, β; then in the corresponding coordinate plane these points read ($\cos\alpha$, $\sin\alpha$), ($\cos\beta$, $\sin\beta$) and the scalar product of the corresponding radii-vectors yields

$$1.1\quad \cos(\beta - \alpha) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad \text{i.e.}$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\,.$$

From this equality other trigonometrical formulae are deducible, like the formula for $\cos(\alpha + \beta)$, $\sin(\alpha + \beta)$ etc.

18. *Scalar components of a vector in coordinate space.*

Let $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$ be 2 points in space; then one has

$$\overrightarrow{OA} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3\,, \qquad \overrightarrow{OB} = b_1\vec{e}_1 + b_2\vec{e}_2 + b_3\vec{e}_3$$

and

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} + \overrightarrow{AO} = \overrightarrow{OB} - \overrightarrow{OA} = b_1\vec{e}_1 + \ldots - a_1\vec{e}_1 - \ldots$$

$$\overrightarrow{AB} = (b_1 - a_1)\vec{e}_1 + (b_2 - a_2)\vec{e}_2 + (b_3 - a_3)\vec{e}_3\,.$$

i.e. *the scalar components of the vector $\overrightarrow{AB}$ are differences of the components of the final point and of the initial point.*

19. *Square of the distance of 2 points. Sphere. Circle.*

$$AB^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2\,.$$

In particular the sphere-border $S(A = r)$ as the set of all the points $P = (x, y, z)$ whose distance from A is r is represented by the condition $PA^2 = r^2$ i.e. $(x - a_1)^2 + (y - a_2)^2 + (z - a_3)^2 = r^2$. The intersection of the sphere $S(a, r)$ and the plane $z = 0$ yields the circle-line $(x - a_1)^2 + (y - a_2)^2 = r^2$.

The corresponding full sphere and full circle should read

$$(x - a_1)^2 + (y - a_2)^2 + (z - a/)^2 \leq r^2$$

$$(x - a_1)^2 + (y - a_2) \leq r^2$$

respectively.

20. *Fundamental problem.*

20. 1. Given a non-null vector $\vec{v}$ and a point $P_0 = (x_0, y_0, z_0)$; determine the plane $\perp \vec{v}$ and containing the point P_0. The requested plane is composed of all the points P such that $\vec{v} \perp \overrightarrow{P_0 P}$ i.e.

$$\vec{v}\,\overrightarrow{P_0 P} = 0 .$$

This is the geometrical equation of the requested plane. In scalar form it yields (writing $P = (x, y, z)$):

$$v_1 (x - x_0) + v_2 (y - y_0) + v_3 (z - z_0) = 0 . \qquad (1)$$

In particular, all radius-vectors orthogonal to a given radius-vector $\vec{v} \neq \vec{0}$ determine a plane composed of the points (x, y, z) such that

$$v_1 x + v_2 y + v_3 z = 0 .$$

This plane contains the origin (0, 0, 0).

20. 2. Equation (1) is linear in x, y, z with coefficients which are not all zero. Conversely, every such equation

$$ax + by + cz + d = 0 \qquad (2)$$

where at least one of the numbers a, b, c is $\neq 0$ represents a plane $\perp$ on the radiusvector $0 \to (a, b, c)$; if e.g. $c \neq$ o, then the plane contains the point (x_0, y_0, z_0) where x_0, y_0 are any numbers and where z_0 is determined by the condition

$$ax_0 + by_0 + cz_0 + d = 0 .$$

20. 3. Consequently, *to every plane (2) is associated a vector which is orthogonal to it*; we could norm it to have the length $= 1$, its components are then $(a \,.\, (a^2 + b^2 + c^2)^{-1/2}$, $b\,(a^2 + b^2 + c^2)^{-1/2}$, $c\,(\quad)^{-1/2})$, where the sign of $(\quad)^{-1/2}$ is $+$ or $-$; one could chose the sign so that the last of the non vanishing components of this vector be > 0.

21. *Parallelism and orthogonality.*

21. 1. The prototype of parallel sets are *opposite sides* in a parallelogram.

21. 2. The lines bearing these edges are defined to be parallel. Moreover, any line is considered to be parallel with itself.

21. 3. If a, b are straight lines and if $a \parallel b$ (a is parallel to b), then any finite or infinite interval of a is considered to be parallel with b as well as with each ray or interval on b. In particular, if $\vec{x}$ is a vector located on a, then $\vec{x} \parallel b$ as well as $\vec{x} \parallel \vec{y}$ for any vector $\vec{y}$ which is located on b.

21. 4. *Parallel planes* are such ones which have no common point. A straight line l and a plane are said to be parallel provided their common part is void or provided l lies in the plane.

21. 5. The parallelism of a vector and a plane is defined in an obvious way.

21. 6. *We postulate the transitivity of the relation of parallelism: if* a $\parallel$ b *and* b $\parallel$ c *then* a $\parallel$ c.

21. 7. When one defines exactly the space, the planes, the lines etc. then these properties are provable. *The space is defined as the set of all the ordered triples* (x, y, z) *of real numbers*, the distance is defined as the non negative antisquare of the sum of squares of differences etc. The space is definable also as the set of all radius-vectors (x, y, z) etc.

21. 8. *Orthogonality of two vectors* $\vec{\mathrm{v}} = (\mathrm{a, b, c})$, $\vec{\mathrm{v}}' = (\mathrm{a', b', c'})$ is expressed by the relation $aa' + bb' + \mathrm{cc}' = 0$ i.e. $\vec{v}\vec{v}' = 0$.

21. 9. *Parallelism of two non-null vectors* $\vec{\mathrm{v}} = (\mathrm{a, b, c})$, $\vec{\mathrm{v}}' = (a', b', c')$ is expressed by the equality $\vec{v}' = k\vec{v}$ i.e. $a' = ka$, $b' = kb$, $c' = kc$, where k is a number.

21. 10. *If* $\vec{\mathrm{v}} \perp \vec{\mathrm{u}}$, $\vec{\mathrm{u}} \parallel \vec{\mathrm{t}}$, *then* $\vec{\mathrm{v}} \perp \vec{\mathrm{t}}$.

As a matter of fact one has by hypothesis $\vec{v}\vec{u} = 0$, $\lambda \neq 0$, $\vec{u} = \lambda\vec{t}$ i.e.

$$\vec{v}\,(\lambda\vec{t}) = 0 \Rightarrow \lambda\vec{v}\vec{t} = 0 \Rightarrow \vec{v}\vec{t} = 0 \Rightarrow \vec{v} \perp \vec{t}.$$

21. 11. *How to express the parallelism (normality) of a vector* $\vec{\mathrm{v}}$ *and a plane* p ? It is expressed as the orthogonality (parallelism) of this vector $\vec{v}$ and the normal vector n (p) attached

to the plane p:

$$\vec{v} \parallel p \iff \vec{v} \perp n(p) \quad \text{i.e.} \quad \vec{v}n(p) = 0$$

$$\vec{v} \perp p \iff v = k \cdot n(p) .$$

If $\vec{v} = (v_1, v_2, v_3)$ and if the equation of the plane is $Ax + By + Cz + D = 0$, then $n(p) = (A, B, C)$ and consequently:

$$\vec{v} \parallel p \iff v_1 A + v_2 B + v_3 C = 0$$

$$\vec{v} \perp p \iff \frac{v_1}{A} = \frac{v_2}{B} = \frac{v_3}{C} .$$

22. *Some extremality considerations.*

Union and intersection of sets (cf. § 3. 4).

Interval (§ 4. 1) — angle. Triangular relation. Inscribed and circumscribed circles (cf. § 8. 2). Distance between a point and a line or a circle-line, 2 lines (in space). Angle between a line and a plane or between 2 planes. Isosceles triangle with given basis and perimeter as greatest triangle with these data (occuring of ellipse) etc.

23. *Measure geometry* (perimeter, area and volume calculations) are to be taken in the usual extent. *Experimental work and measurations should be stressed.*

24. *Conics* are to be introduced as sections of cones and planes. From the analytical point of view, ellipse and hyperbola are introduced by their plane properties; the properties of the hyperbola are obtainable from those of an ellipse by considering the hyperbola as an ellipse with imaginary axis.

The parabola is to be introduced and examined as graph of quadratic functions.

Use is made of differential quotient for slope of a tangent.

25. *Trigonometry* is backed upon the number circle line, addition theorem for cosine and cosine theorem. Other items are deducible from them.

26. *Elements of descriptive geometry* are to be taken too.

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