ON THE ZEROS OF BERNOULLI POLYNOMIALS OF EVEN ORDER

Autor(en): **Ostrowski, A. M.**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **6 (1960)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-36334

ON THE ZEROS OF BERNOULLI POLYNOMIALS OF EVEN ORDER

by A. M. OSTROWSKI [1])

(Recu le 12 avril 1960)

§ 1. It is well known that the Bernoulli polynomial $B_{2\nu}(x)$ has exactly one zero r_ν between 0 and $\frac{1}{4}$. NÖRLUND [2], p. 131, proves that r_ν lies between $\frac{1}{6}$ and $\frac{1}{4}$ and states that r_ν tends to $\frac{1}{4}$ as $\nu \to \infty$.

In what follows we shall prove that r_ν tends *monotonically* to $\frac{1}{4}$. More precisely we have

$$\frac{\frac{1}{4} - r_1}{\frac{1}{4} - r_2} > 4 > \frac{\frac{1}{4} - r_\nu}{\frac{1}{4} - r_{\nu+1}} > 3 \qquad (\nu = 2, 3, \ldots) . \tag{1}$$

This follows from the relation

$$\theta_\nu \overset{\text{def}}{=} 2\pi\left(\frac{1}{4} - r_\nu\right) = \frac{1}{2^{2\nu}} - \frac{1}{4^{2\nu}} + \frac{4}{6^{2\nu}} - \frac{17}{6}\frac{1}{8^{2\nu}} - \frac{4}{10^{2\nu}} - \frac{4}{12^{2\nu}} + \\ + \frac{13\,p}{14^{2\nu}}, \; 0 < p < 1 \; (\nu = 1, 2, \ldots) . \tag{2}$$

We obtain (2) from the relation derived in sec. 12 of this paper:

$$\sin\theta_\nu = \frac{1}{2^{2\nu}} - \frac{1}{4^{2\nu}} + \frac{4}{6^{2\nu}} - \frac{3}{8^{2\nu}} - \frac{4}{10^{2\nu}} - \frac{4}{12^{2\nu}} + \\ + \frac{13\,p}{14^{2\nu}}, \; \frac{1}{40} \leqq p < 1 \quad (\nu = 1, 2, \ldots) . \tag{3}$$

---

[1]) The preparation of this paper was sponsored by the U.S. Army under Contract No. DA-11-022-ORD-2059, Mathematics Research Center, U.S. Army. I am indebted for discussions to Mr. H. Fishman, M.S., and Dr. J. C. P. Miller, to the latter particularly also for checking many computations.

In 1940 D. H. LEHMER [1] proved that

$$\theta_\nu < \frac{1}{2^{2\nu}} \quad (4)$$

and

$$\theta_\nu = \frac{1}{2^{2\nu}} - \frac{1}{4^{2\nu}} + \frac{4}{6^{2\nu}} + O\left(\frac{1}{8^{2\nu}}\right) \qquad (\nu \to \infty) . \quad (5)$$

However, obviously the monotony of the r_ν cannot be derived from an asymptotic ıelation like (5), nor can a relation of this type be used for calculating the r_ν. [1])

In the last part of this paper I discuss the asymptotic development of sin θ_ν into a Dirichlet series with even denominators and integer coefficients. Some numerical computations made by Dr. Miller suggested that these coefficients may present a certain interest from the arithmetical point of view. We prove indeed some congruence properties of these coefficients mod 4 and mod 8. Further, certain sequences of these coefficients can be explicitly determined using Riemann's ζ-function and the Dirichlet's L-function corresponding to the modulus 4.

§ 2. We will write from now on μ for 2ν and use with Lehmer the expression

$$(-1)^{\nu-1} \frac{(2\pi)^\mu}{\mu\,!} B_\mu(x) = \sum_{\varkappa=1}^{\infty} \frac{\cos 2\varkappa\pi x}{\varkappa^\mu} . \quad (6)$$

We have from (2)

$$r_\nu = \frac{1}{4} - \frac{\theta}{2\pi} , \quad (7)$$

where we omit for simplicity sake the index ν of θ. Introducing (7) into (6) we obtain, putting the result $= 0$,

$$\sin\theta + \sum_{\omega=1}^{\infty} (-1)^\omega \frac{\cos 2\omega\theta}{(2\omega)^\mu} + \sum_{\omega=1}^{\infty} (-1)^\omega \frac{\sin(2\omega+1)\theta}{(2\omega+1)^\mu} = 0 . \quad (8)$$

---

1) Our first proof of the monotony of the r_ν was obtained in using the Newton-Raphson approximation. It may be mentioned that for the numerical computation of the values r_ν Dr. Miller found it particularly convenient to derive from the formula (8) in sec. 2 an approximate quadratic equation for sin θ_ν with an error term $O(24^{-2\nu})$.

Put

$$\sigma = \sin\theta\,, \quad \delta = \frac{1}{2^\mu}\,, \tag{9}$$

then we obtain, isolating in our equation the term $\sin\theta = \sigma$,

$$\sigma < \delta\cos 2\theta + \frac{\sin 3\theta}{3^\mu} - \delta^2\cos 4\theta + \sum_{\varkappa=5}^{\infty}\frac{1}{\varkappa^\mu}\,. \tag{10}$$

§ 3. We will have to use repeatedly the inequality

$$\sin mx < m\sin x\left(m > 1\,,\ 0 < mx < \frac{3\pi}{2}\right), \tag{11}$$

which follows easily directly for $m = 2$ and for $m \geqq 3$ from the fact that the expression

$$(\sin mx - m\sin x)' = m(\cos mx - \cos x)$$

is negative under the conditions indicated in (11). It follows then, since

$$\theta = 2\pi\left(\frac{1}{4} - r_\nu\right) < \frac{2\pi}{12} = \frac{\pi}{6} < 0.6\,,$$

$$\sin 3\theta < 3\sigma\,, \quad \cos 4\theta = 1 - 2\sin^2 2\theta > 1 - 8\sigma^2\,.$$

Introducing this into (10) and using $\cos 2\theta = 1 - 2\sigma^2$ we obtain

$$\sigma < \delta(1 - 2\sigma^2) + \frac{3\sigma}{3^\mu} - \delta^2(1 - 8\sigma^2) + \sum_{\varkappa=5}^{\infty}\frac{1}{\varkappa^\mu}\,. \tag{12}$$

We will assume from now on that we have

$$\nu \geqq 5\,, \quad \mu \geqq 10\,. \tag{13}$$

This assumption will be dropped only in the section 12. — Then we have

$$\frac{1}{\delta^2}\sum_{\varkappa=5}^{\infty}\frac{1}{\varkappa^\mu} = \sum_{\varkappa=5}^{\infty}\left(\frac{4}{\varkappa}\right)^\mu \leq \sum_{\varkappa=5}^{\infty}\left(\frac{4}{\varkappa}\right)^6 = 4^6\left(\sum_{\varkappa=1}^{\infty}\frac{1}{\varkappa^6} - 1 - \frac{1}{2^6} - \frac{1}{3^6} - \frac{1}{4^6}\right)$$

$$= 4^6\left(1.01734306 - 1 - \frac{1}{2^6} - \frac{1}{3^6} - \frac{1}{4^6}\right) < 0.42\,.$$

We introduce this into (12), bring the second term to the left and divide by δ. Thus we obtain

$$\left(1 - \frac{3}{3^{\mu}}\right)\sigma/\delta < 1 - \sigma^2 (2 - 8\,\delta) - 0.58\delta < 1 - 0.58\,\delta .$$

On the other hand, we have from $\nu = 5$ on

$$3\left(\frac{2}{3}\right)^{\mu} \leqq \frac{2^{10}}{3^9} = 0.052 \ldots < \frac{0.08}{1 - \delta/2},$$

$$\frac{3}{3^{\mu}} < \frac{0.08\,\delta}{1 - \delta/2}, \quad 1 - \frac{3}{3^{2\nu}} > 1 - \frac{0.08\,\delta}{1 - \delta/2} = \frac{1 - 0.58\,\delta}{1 - \delta/2},$$

and it follows

$$\frac{\sigma}{\delta} < 1 - \delta/2, \quad \sigma < \delta - \frac{\delta^2}{2}. \tag{14}$$

As the arc sin series,

$$\arcsin x = x + \frac{x^3}{6} + \sum_{\nu=2}^{\infty} \frac{1 \,.\, 3 \ldots (2\nu - 1)}{2 \,.\, 4 \ldots (2\nu)} \frac{x^{2\nu+1}}{2\nu + 1},$$

has monotonically decreasing positive coefficients, we have easily for $0 < x < 1$ and convenient p with $o < p < 1$:

$$\arcsin x = x + \frac{x^3}{6} + \frac{3px^5}{40} \frac{1}{1 - x^2} \tag{15}$$

For $x = \sigma = \sin\theta < \delta = \frac{1}{2^{2\nu}}$ it follows

$$\theta < \sigma\left[1 + \frac{\sigma}{6\,(1 - \sigma^2)}\,\sigma\right] <$$

$$< \delta\left(1 - \frac{\delta}{2}\right)\left[1 + \frac{\sigma}{2}\right] < \delta\left(1 - \frac{\delta}{2}\right)\left(1 + \frac{\delta}{2}\right),$$

$$\theta < \delta, \tag{16}$$

which proves Lehmer's inequality $\theta < \delta$ for $\nu \geqq 5$.

§ 4. We rewrite now (8) in the form

$$\sigma = \sum_{\omega=1}^{6} (-1)^{\omega+1} \frac{\cos 2\omega\theta}{(2\omega)^{\mu}} + \sum_{\omega=1}^{5} (-1)^{\omega+1} \frac{\sin (2\omega + 1)\,\theta}{(2\omega + 1)^{\mu}} + S, \tag{17a}$$

$$S = \sum_{\omega=7}^{\infty} (-1)^{\omega+1} \frac{\cos 2\omega\theta}{(2\omega)^{\mu}} + \sum_{\omega=6}^{\infty} (-1)^{\omega+1} \frac{\sin (2\omega + 1)\,\theta}{(2\omega + 1)^{\mu}}. \tag{17b}$$

We shall express the remainder terms in multiples of

$$m = 14^{-\mu} . \qquad (18)$$

For any integer $n > 14$ we have by (13)

$$n^{-\mu}/m = (14/n)^{\mu} \leqq (14/n)^{10}$$

and therefore

$$\frac{1}{16^{\mu}} < 0.27\, m \,, \; \frac{1}{18^{\mu}} < 0.082\, m \,, \; \frac{1}{20^{\mu}} < 0.032\, m \,, \; \frac{1}{24^{\mu}} < 0.005\, m \,. \qquad (19)$$

In our estimates we shall use the inequality

$$\frac{\sin x\alpha}{x^s} > \frac{\sin (x+1)\alpha}{(x+1)^s} \quad (s \geqq 1\,, \quad x > 0\,, \quad (x+1)\alpha < \pi)\,. \qquad (20)$$

To prove this inequality, observe that it is sufficient to prove

$$\frac{x+1}{x} > \frac{\sin (x+1)\alpha}{\sin x\alpha}\,,$$

that is,

$$(x+1) \sin x\alpha - x \sin (x+1)\alpha > 0\,.$$

But the expression to the left is $= 0$ for $\alpha = 0$ and its derivative with respect to α,

$$(x+1)\, x\, (\cos x\alpha - \cos (x+1)\alpha)$$

is > 0 under the conditions of (20).

§ 5. We use now in the first sum of (17*b*) the formula

$$\cos 2\omega\theta = 1 - 2 \sin^2 \omega\theta$$

and decompose S as follows

$$S = \sum_{\omega=7}^{\infty} \frac{(-1)^{\omega+1}}{(2\omega)^{\mu}} - 2 \sum_{\omega=7}^{\infty} (-1)^{\omega+1} \frac{\sin^2 \omega\theta}{(2\omega)^{\mu}} +$$

$$+ \sum_{\omega=6}^{\infty} (-1)^{\omega+1} \frac{\sin (2\omega+1)\theta}{(2\omega+1)^{\mu}}\,. \qquad (21)$$

Here we have by (18) and (19):

$$m > \sum_{\omega=7}^{\infty} \frac{(-1)^{\omega+1}}{(2\omega)^{\mu}} > m - \frac{1}{16^{\mu}} \geqq m\,(1 - (14/16)^{10}) > (1 - 0.27)\, m\ .$$

In what follows, we will use the letter p, with or without indices, to denote positive numbers < 1, which need not be otherwise specified and need not be the same.

We can therefore write

$$\sum_{\omega=7}^{\infty} \frac{(-1)^{\omega+1}}{(2\omega)^{\mu}} = (1 - 0.27\, p)\, m\ . \tag{22}$$

Further, we have, using (11) and (14),

$$2\left|\sum_{\omega=7}^{\infty} (-1)^{\omega+1} \frac{\sin^2 \omega\theta}{(2\omega)^{\mu}}\right| < \frac{1}{2^{3\mu-1}} \sum_{\omega=7}^{1000} \frac{1}{\omega^{\mu-2}} + \frac{1}{2^{\mu-1}} \sum_{\omega=1001}^{\infty} \frac{1}{\omega^{\mu}} <$$

$$< \frac{1}{2^{3\mu-1}} \int_6^{\infty} \frac{dx}{x^{\mu-2}} + \frac{1}{2^{\mu-1}} \int_{1000}^{\infty} \frac{dx}{x^{\mu}} < \frac{432}{7}\, \frac{1}{8^{\mu}\,.\,6^{\mu}} + \frac{2000}{9.2000^{\mu}} <$$

$$< \frac{100}{48^{\mu}} < 100 \left(\frac{7}{24}\right)^{10} m < 100 \left(\frac{50}{500}\right)^{5} m < 0.01\, m\ .$$

And, again, by (11) and (19)

$$\left|\sum_{\omega=6}^{\infty} (-1)^{\omega+1} \frac{\sin (2\omega+1)\theta}{(2\omega+1)^{\mu}}\right| < \delta \sum_{\omega=6}^{\infty} \frac{1}{(2\omega+1)^{\mu-1}} < \frac{2}{2^{4\nu}} \sum_{\omega=6}^{\infty} \frac{1}{\omega^{\mu-1}} <$$

$$< \frac{2}{4^{\mu}} \int_5^{\infty} \frac{dx}{x^{\mu-1}} < \frac{25}{8.4^{\mu}} \cdot \frac{1}{5^{\mu}} = \frac{25}{8}\, \frac{1}{20^{\mu}} < 0.1\, m\ .$$

It follows now from (21) and (22)

$$S = (1 - 0.38\, p)\, m\ . \tag{23}$$

§ 6. We consider the second sum in (17*a*). We have by (14) and (20)

$$\frac{\sin 7\theta}{7^{\mu}} - \frac{\sin 9\theta}{9^{\mu}} + \frac{\sin 11\theta}{11^{\mu}} = p\, \frac{\sin 7\theta}{7^{\mu}} = p'\, \frac{7\sigma}{7^{\mu}} = 7\, pm\ .$$

Further, using $\sin 3\theta = 3\sigma - 4\sigma^3$, $\sin 5\theta = 5\sigma - 20\sigma^3 + 16\sigma^5$,

$$\frac{\sin 3\theta}{3^\mu} - \frac{\sin 5\theta}{5^\mu}\left(\frac{3}{3^\mu} - \frac{5}{5^\mu}\right)\sigma = -4\sigma^3\left[\frac{1}{3^\mu} - \frac{5}{5^\mu} + 4\frac{\sigma^2}{5^\mu}\right] =$$

$$= -\frac{4p}{24^\mu} = -0.02\,pm\,.$$

Therefore

$$\sum_{\omega=1}^{5}(-1)^{\omega+1}\frac{\sin(2\omega+1)\theta}{(2\omega+1)^\mu} = \left(\frac{3}{3^\mu} - \frac{5}{5^\mu}\right)\sigma + (7p - 0.02p')\,m\,. \qquad (24)$$

§ 7. Consider now the first sum in (17*a*). We have

$$\sum_{\omega=1}^{6}(-1)^{\omega+1}\frac{\cos 2\omega\theta}{(2\omega)^\mu} - \left(\frac{1}{2^\mu} - \frac{1}{4^\mu} + \frac{1}{6^\mu} - \frac{1}{8^\mu} + \frac{1}{10^\mu} - \frac{1}{12^\mu}\right) =$$

$$= -2\left(\frac{\sigma^2}{2^\mu} - \frac{\sin^2 2\theta}{4^\mu} + \frac{\sin^2 3\theta}{6^\mu} - \frac{\sin^2 4\theta}{8^\mu} + \frac{\sin^2 5\theta}{10^\mu} - \frac{\sin^2 6\theta}{12^\mu}\right) =$$

$$= -2\delta\sigma^2 + S_1\,,$$

$$S_1 = 2\sum_{\omega=2}^{6}(-1)^\omega\frac{\sin^2\omega\theta}{(2\omega)^\mu}\,,$$

where by (20) and (19)

$$S_1 = 2p\frac{\sin^2 2\theta}{4^\mu} = \frac{8p'\sigma^2}{4^\mu} = \frac{8p''}{16^\mu} = 2.2\,pm\,.$$

We have, therefore,

$$\sum_{\omega=1}^{6}(-1)^{\omega+1}\frac{\cos 2\omega\theta}{(2\omega)^\mu} = \sum_{\omega=1}^{6}\frac{(-1)^{\omega+1}}{(2\omega)^\mu} - 2\delta\sigma^2 + 2.2\,pm\,. \qquad (25)$$

We have now by (23), (24) and (25) from (17*a*)

$$\sigma = \sum_{\omega=1}^{6}\frac{(-1)^{\omega+1}}{(2\omega)^\mu} + \left(\frac{3}{3^\mu} - \frac{5}{5^\mu}\right)\sigma - 2\delta\sigma^2 + (1 - 0.4p + 9.2p')m\,. \qquad (26)$$

Here the last term can be written as

$$m\,(0.6 + 9.2p + 0.4\,(1 - p')) = (0.6 + 9.6p'')\,m\,.$$

§ 8. We put now, using (14),

$$\sigma = \delta - \eta\,, \quad 0 < \eta < \delta\,. \tag{27}$$

Then, (26) becomes

$$\delta - \eta = \sum_{\omega=1}^{6} \frac{(-1^{\omega+1}}{(2\omega)^{\mu}} + \frac{3}{6^{\mu}} - \frac{5}{10^{\mu}} - \left(\frac{3}{3^{\mu}} - \frac{5}{5^{\mu}}\right)\eta - 2\delta(\delta^2 - 2\delta\eta + \eta^2) +$$
$$+ (0.6 + 9.6p)\,m\,.$$

Here we bring all terms containing η to the right:

$$\delta + \sum_{\omega=1}^{6} \frac{(-1)^{\omega}}{(2\omega)^{\mu}} - \frac{3}{6^{\mu}} + \frac{5}{10^{\mu}} + 2\delta^3 - (0.6 + 9.6p)\,m =$$
$$= \eta\left(1 - \frac{3}{3^{\mu}} + \frac{5}{5^{\mu}}\right) + 4\delta^2\eta - 2\delta\eta^2\,.$$

We put now

$$x = \frac{3}{3^{\mu}} - \frac{4}{4^{\mu}} - \frac{5}{5^{\mu}}\,. \tag{28}$$

Then we obtain

$$(1 - x)\,\eta - 2\delta\eta^2 = \frac{1}{4^{\mu}} - R + E\,, \tag{29}$$

where we have put

$$R = \frac{4}{6^{\mu}} - \frac{3}{8^{\mu}} - \frac{4}{10^{\mu}} - \frac{1}{12^{\mu}}\,, \tag{30}$$

$$E = -(0.6 + 9.6p)\,m\,. \tag{31}$$

§ 9. The expression x defined by (28) is *positive* as follows from

$$1 = \frac{1}{2} + \frac{1}{2} > \left(\frac{3}{4}\right)^9 + \left(\frac{3}{5}\right)^9 \geqq \left(\frac{3}{4}\right)^{\mu-1} + \left(\frac{3}{5}\right)^{\mu-1}\,.$$

In the same way, we see that R is *positive*:

$$4 > 1 + 1 + 1 > 3\left(\frac{3}{4}\right)^{10} + 4\left(\frac{3}{5}\right)^{10} + \left(\frac{1}{2}\right)^{10}\,.$$

We have, therefore,

$$0 < x < \frac{3}{3^\mu} < \frac{1}{1001}, \quad \frac{1}{1-x} < 1 + \frac{1}{1000},$$

$$R < \frac{4}{6^\mu}, \quad xR = \frac{12p}{18^\mu} = 12.0.082\,p'm = 0.99\,p''m,$$

$$\frac{xR}{1-x} = pm. \tag{32}$$

Further, by (19),

$$\delta^2 x^2 = \frac{9}{36^\mu} \leqq 9.2^{-10}(18)^{-\mu} = 0.01\,pm,$$

$$\frac{\delta^2 x^2}{1-x} = 0.02\,pm. \tag{33}$$

The expression to the right in (29) is, as $R > 0$, $< \delta^2$, and we have from (29) by (27)

$$\left(1 - \frac{1}{1001}\right)\frac{\eta}{\delta^2} < 2\delta\left(\frac{\eta}{\delta}\right)^2 + 1 < 1.002, \quad \eta < 1.04\,\delta^2,$$

$$\eta^2 < 1.1\,\delta^4, \quad 2\delta\eta^2 < 3\delta^5 = 3\left(\frac{7}{16}\right)^{10} pm = 0.1\,(1-p')\,m.$$

Using this in (29), we obtain now

$$(1-x)\,\eta = \delta^2 - R + E_1, \tag{34}$$

$$E_1 = -(0.5 + 9.7p)\,m. \tag{35}$$

§ 10. From (35), we have now easily

$$|E_1| < 11\,m,$$

$$\left|\frac{E_1 x}{1-x}\right| < 0.05m. \tag{36}$$

Dividing now (34) by $1-x$ we have

$$\eta - \delta^2 + R - \delta^2 x = \frac{\delta^2 x^2}{1-x} - \frac{Rx}{1-x} + E_1 + \frac{E_1 x}{1-x}, \tag{37}$$

and from (32), (33), (35) and (36), we have for the right-hand expression in (37):

$$(0.02p_1 - p_2 + 0.05p_3 - 0.5 - 9.7p_5 - 0.05p_4) = -(0.4 + 10.9)\,pm.$$

On the other hand, we have from (28) by (19)

$$\delta^2 x - \frac{3}{12^\mu} = -\left(\frac{4}{16^\mu} + \frac{5}{20^\mu}\right) = -\left(4\left(\frac{7}{8}\right)^{10} + 5\left(\frac{7}{10}\right)^{10}\right) pm .$$

$$\delta^2 x = \frac{3}{12^\mu} - 1.4\, pm .$$

Introducing this into (37), we finally obtain

$$\eta = \frac{1}{4^\mu} - \frac{4}{6^\mu} + \frac{3}{8^\mu} + \frac{4}{10^\mu} + \frac{4}{12^\mu} - \frac{0.4 + 12.3p}{14^\mu} . \qquad (38)$$

By (27), we have then

$$\sigma = \sin\theta = \frac{1}{2^\mu} - \frac{1}{4^\mu} + \frac{4}{6^\mu} - \frac{3}{8^\mu} - \frac{4}{10^\mu} - \frac{4}{12^\mu} + \frac{13p}{14^\mu}, \qquad (39)$$

$$\frac{1}{40} \leqq p < 1 \ (\nu \geqq 5).$$

From (38), it now follows easily that

$$\eta < \delta^2 . \qquad (40)$$

§ 11. We shall apply now (15) to $x = \sigma$. We have obviously by (14) and (19)

$$\frac{\sigma^5}{1-\sigma^2} < \frac{\delta^5}{1-\delta^2} < \frac{1}{16^\mu}\,\frac{1}{1-10^{-6}} \cdot \frac{1}{2^\mu} < \frac{0.001}{16^\mu} < \frac{2.7}{10^4} m ,$$

$$\frac{3}{40}\,\frac{\sigma^5}{1-\sigma^2} < \frac{3m}{10^5} .$$

Further, from (27), (40) and (19)

$$\delta^3 - \sigma^3 = 3\,\delta^2\eta - 3\,\delta\,\eta^2 + \eta^3 < 3\,\delta^2\eta < 3\,\delta^4 < 0.9m ,$$

and, therefore, from (15) and (39), as $\frac{0.9}{6} < \frac{13}{40}$,

$$\theta = \frac{1}{2^\mu} - \frac{1}{4^\mu} + \frac{4}{6^\mu} - \frac{17}{6}\,\frac{1}{8^\mu} - \frac{4}{10^\mu} - \frac{4}{12^\mu} + \frac{13p}{14^\mu} . \qquad (41)$$

§ 12. The formulae (39) and (41) have been only derived for $\nu \geq 5$. However, the direct comparison of the expressions given by these formulae with the values of r_ν for $\nu = 1, 2, 3, 4$ given with ten decimals by Lehmer [1]) shows that these formulae also hold for $\nu = 1, 2, 3, 4$. This proves the formulae (2) and (3).

§ 13. Lehmer's formula (5) and our formulae (2), (3) suggest that θ, as well as $\sin\theta$, possess asymptotic development by infinite Dirichlet series. In this connection, apparently, the development of $\sin\theta$ gives a more natural and more interesting result. We will now prove that there exists a Dirichlet series

$$\sum_{n=1}^{\infty} \frac{q_n}{(2n)^\mu} \tag{42}$$

with *integer* q_n such that we have for $\nu \to \infty$ and every positive integer N

$$\eta_N \overset{\text{def}}{=} \sin\theta_\nu - \sum_{n=1}^{N} \frac{q_n}{(2n)^\mu} = O\left((2N+2)^{-\mu}\right) (\nu \to \infty) \ldots \tag{43}$$

§ 14. It is obvious from (3) that (43) is true for $N \leq 6$ for a certain Dirichlet polynomial. Assume that we have for a certain value N the relation (43) for a certain Dirichlet polynomial with N terms. Introducing (8) into (43), we have then

$$\eta_N = \sum_{\omega=1}^{2N} (-1)^{\omega+1} \left(\frac{\cos 2\omega\theta_\nu}{(2\omega)^\mu} + \frac{\sin(2\omega+1)\theta_\nu}{(2\omega+1)^\mu}\right) - \sum_{\omega=1}^{N} \frac{q_\omega}{(2\omega)^\mu} + $$
$$+ O\left(\frac{1}{(4N+2)^\mu}\right). \tag{44}$$

Here both $\cos 2\omega\theta$ and $\sin(2\omega+1)\theta$ are polynomials in $\sigma_\nu = \sin\theta_\nu$ *with integer coefficients.* Putting these polynomials in (44), we obtain then, denoting by M the greatest of their degrees,

$$\cos 2\omega\theta_\nu = \sum_{\varkappa=0}^{M} a_\varkappa \sigma_\nu^\varkappa = \sum_{n=1}^{2N} \frac{A_n^{(\omega)}}{(2n)^\mu} + O\left(\frac{1}{(4N+2)^\mu}\right),$$

[1]) These values have been checked independently of Lehmer by Dr. J. C. P. Miller.

$$\sin(2\omega+1)\,\theta_\nu = \sum_{\varkappa=1}^{M} b_\varkappa\,\sigma_\nu^\varkappa = \sum_{n=1}^{2N} \frac{B_n^{(\omega)}}{(2n)^\mu} + O\left(\frac{1}{(4N+2)^\mu}\right),$$

$$\frac{\cos 2\omega\theta_\nu}{(2\omega)^\mu} + \frac{\sin(2\omega+1)\,\theta_\nu}{(2\omega+1)^\mu} = \sum_{n=1}^{2N} \frac{C_n^{(\omega)}}{(2n)^\mu} + O\left(\frac{1}{(4N+2)^\mu}\right),$$

with integers $A_n^{(\omega)}$, $B_n^{(\omega)}$ and $C_n^{(\omega)}$, and therefore

$$\eta_N = \sum_{n=1}^{2N} \frac{D_n}{(2n)^\mu} + O\left(\frac{1}{(4N+2)^\mu}\right) \tag{45}$$

with integer D_N. But, now it follows from (43) that

$$D_1 = D_2 = \ldots = D_N = 0,$$

and we have, therefore, putting

$$q_n = D_n \ (N < n \leqq 2N), \tag{46}$$

again the relation (43) with 2N instead of N.

Repeating this procedure indefinitely, the existence of the Dirichlet series (42) is proved.

Introducing the asymptotic development (42) into the Maclaurin series for arcsin x, we obtain an asymptotic Dirichlet development of θ_ν itself. However, in this development, as we see from (2), the coefficients are no longer integers.

§ 15. In what follows, we give with the kind permission of Dr. J. C. P. Miller the first fifty coefficients of the series (42) which he has computed.

n	1	2	3	4	5	6	7	8	9	10
q_n	1	−1	4	−3	−4	−4	8	11	4	4
n	11	12	13	14	15	16	17	18	19	20
q_n	12	−48	−12	−8	−16	25	−16	−4	20	0
n	21	22	23	24	25	26	27	28	29	30
q_n	32	−12	24	248	−4	12	4	−208	−28	16
n	31	32	33	34	35	36	37	38	39	40
q_n	32	−41	48	16	−32	−400	−36	−20	−48	88
n	41	42	43	44	45	46	47	48	49	50
q_n	−40	−32	44	−544	−16	−24	48	732	8	4

Dr. Miller drew my attention to the properties of the coefficients of the series (42) which appear to be suggested by the above values. The odd coefficients correspond exactly to the denominators which are powers of 2, while all other numerators are divisible by 4.

We will now prove that these properties are indeed true in the general case. Beyond that, we will prove for the numerators q_n which correspond to $n = 2^k$, that we have

$$q_n \equiv (-1)^k \equiv 2k + 1 \pmod 4 \quad (n = 2^k) . \tag{47}$$

We will prove even a more precise formula

$$q_n \equiv (-1)^{k-1}\, 2\,(k - 1) - 1 \equiv 4k^2 + 2k + 1 \pmod 8 \; (n = 2^k) . \tag{48}$$

Further, we will determine directly all q_n corresponding to n non divisible by 8 by forming generating Dirichlet series for these numerators. We will find in particular for an odd natural u

$$q_{2u} = - q_u \; (u \equiv 1 \pmod 2)) . \tag{49}$$

§ 16. Expressing $\sin (2\omega + 1)\,\theta$ and $\cos 2\omega\theta$ as polynomials in $\sigma = \sin \theta$, we have, putting $2\omega + 1 = u$, $2\omega = g$,

$$\sin u\,\theta = u\,\sigma + R_n(\sigma) , \tag{50}$$

$$\cos g\,\theta - 1 = - g^2 \frac{\sigma^2}{2} + T_g(\sigma) , \tag{51}$$

where our $R_u(\sigma)$ and $T_g(\sigma)$ are polynomials in σ with integer coefficients, which are all *multiples of* 4.

Indeed, this is true for $\omega = 1$. Assuming our assertions true for a certain ω, use the following relations and congruences mod 4:

$$\cos (2\omega + 2)\,\theta - \cos 2\omega\theta = - 2 \sin \theta \sin (2\omega + 1)\,\theta \equiv - 2\sigma\,(2\omega + 1)\,\sigma$$
$$\equiv - 2\sigma^2 \pmod 4 ,$$

$$\cos (2\omega + 2)\,\theta \equiv 1 - 2\omega^2\,\sigma^2 - 2\sigma^2 \equiv 1 - 2\,(\omega + 1)^2\,\sigma^2 \pmod 4 ,$$

which proves our assertion (51) for $\omega + 1$, and

$$\sin (2\omega + 3)\,\theta - \sin (2\omega + 1)\,\theta = 2\sigma \cos 2\,(\omega + 1)\,\theta$$
$$\equiv 2\sigma\,(1 - 2\,(\omega + 1)^2\,\sigma^2) \equiv 2\sigma \pmod 4 ,$$

which proves our assertion (50) for $\omega + 1$.

Further, it follows from the identity with a natural k:

$$\sin 2^k \theta = 2^k \sin \theta \prod_{\nu=1}^{k} \cos 2^{k-\nu} \theta ,$$

that, developing $\cos 2^{k+1} \theta$ in powers of $\sigma = \sin \theta$, we have

$$\cos 2^{k+1} \theta = 1 - 2^{2k+1} \sigma^2 \prod_{\nu=1}^{k} \cos^2 2^{k-\nu} \theta \equiv 1 \pmod{2^{2k+1}} . \qquad (51^0)$$

§ 17. Introducing the relations (50) and (51) into (8) and solving with respect to $\sigma = \sin \theta$, we obtain

$$\sigma = \sum_{\omega=1}^{\infty} \frac{(-1)^{\omega+1}}{(2\omega)^{\mu}} - \sigma \sum_{\omega=1}^{\infty} \frac{(-1)^{\omega}}{(2\omega+1)^{\mu-1}} + 2\sigma^2 \sum_{\omega=1}^{\infty} \frac{(-1)^{\omega}\omega^2}{(2\omega)^{\mu}} -$$

$$- \sum_{\omega=1}^{\infty} (-1)^{\omega} \left(\frac{T_{2\omega}(\sigma)}{(2\omega)^{\mu}} + \frac{R_{2\omega+1}(\sigma)}{(2\omega+1)^{\mu}} \right) . \qquad (52)$$

We introduce now the two Dirichlet series well known in the analytical theory of numbers of which the first is not very different from Riemann's ζ-function while the second is Dirichlet's L-function corresponding to the modulus 4:

$$U(s) = \left(1 - \frac{1}{2^s}\right) \zeta(s) = \sum_{u} \frac{1}{u^s} , \qquad (53)$$

where the summation index u here and in what follows runs through all positive odd integers subject to the restrictions explicitly indicated, and

$$L(s) = \sum_{u} \frac{(-1)^{\frac{u-1}{2}}}{u^s} = \prod_{p>2} \frac{1}{1 - \frac{(-1)^{\frac{p-1}{2}}}{p^s}} , \qquad (54)$$

where p runs through all odd primes. Then we have for the first three right hand terms in (52), putting

$$x = \frac{1}{2^{\mu}} : \qquad (55)$$

$$\sum_{\omega=1}^{\infty} \frac{(-1)^{\omega+1}}{(2\omega)^{\mu}} = x\mathrm{U}(\mu) - \sum_{\varkappa=2}^{\infty} x^{\varkappa}\mathrm{U}(\mu) = \frac{x-2x^2}{1-x}\mathrm{U}(\mu), \tag{56}$$

$$\sum_{\omega=1}^{\infty} \frac{(-1)^{\omega}}{(2\omega+1)^{\mu-1}} = \mathrm{L}(\mu-1) - 1, \tag{57}$$

$$\sum_{\omega=1}^{\infty} \frac{(-1)^{\omega+1}\omega^2}{(2\omega)^{\mu}} = \frac{1}{4}\sum_{\omega=1}^{\infty} \frac{(-1)^{\omega+1}}{(2\omega)^{\mu-2}} =$$

$$= x\mathrm{U}(\mu-2) - \frac{4x^2}{1-4x}\mathrm{U}(\mu-2) = \frac{x-8x^2}{1-4x}\mathrm{U}(\mu-2). \tag{58}$$

Introducing these expressions into (52) and bringing the term with σ to the left, we have

$$\mathrm{L}(\mu-1)\sigma = \frac{x-2x^2}{1-x}\mathrm{U}(\mu) - 2\sigma^2\frac{x-8x^2}{1-4x}\mathrm{U}(\mu-2) -$$

$$-\sum_{\omega=1}^{\infty}(-1)^{\omega}\left(\frac{\mathrm{T}_{2\omega}(\sigma)}{(2\omega)^{\mu}} + \frac{\mathrm{R}_{2\omega+1}(\sigma)}{(2\omega+1)^{\mu}}\right). \tag{59}$$

§ 18. We can replace σ in (59) by the series (42) and consider now (59) modulo 4, taking a Dirichlet series $\sum_{n=1} \frac{a_n}{n^{\mu}}$ as $\equiv 0$ (mod 4) if all a_n are $\equiv 0$ (mod 4). Then the last sum in (59) is $\equiv 0$, by what has been proved in Section 16 about (50) and (51). Further, we have mod 4:

$$\mathrm{L}(\mu-1) = \sum_u \frac{(-1)^{\frac{u-1}{2}}u}{u^{\mu}} \equiv \sum_u \frac{1}{u^{\mu}} = \mathrm{U}(\mu),$$

$$\mathrm{U}(\mu-2) = \sum \frac{(2\omega+1)^2}{(2\omega+1)^{\mu}} \equiv \mathrm{U}(\mu) \equiv \mathrm{L}(\mu-1) \pmod 4.$$

It follows, therefore, from (59)

$$\mathrm{L}(\mu-1)\sigma \equiv \left(\frac{x-2x^2}{1-x} - 2x\sigma^2\right)\mathrm{L}(\mu-1) \pmod 4.$$

If we multiply this congruence on both sides by

$$\mathrm{L}^{-1}(\mu-1) = \prod_{p>2}\left(1 - \frac{(-1)^{\frac{p-1}{2}}p}{p^{\mu}}\right),$$

we obtain

$$\sigma \equiv x - x^2 - x^3 - x^4 - \dots - 2x\sigma^2 \pmod{4} . \qquad (60)$$

Since we have $x^{\varkappa} = \frac{1}{(2\cdot 2^{\varkappa-1})^{\mu}}$, we see that for each $n = 2^{\varkappa} u$ with and *odd* $u > 1$ the contributions to the q_n come only from the last right hand term in (60) and are therefore *even.* Putting this into the right hand expression in (60), we see even that $q_n \equiv 0 \pmod{4}$.

Consider now an n of the form $n = 2^k$, $k \geq 1$. The corresponding term in (42) is then $q_n x^{k+1}$. Then the right hand series in (60) contributes -1 to q_n. As to the contribution of $-2x\sigma^2 \pmod{4}$, it is equal to -2 times the coefficient of x^k in σ^2, taken (mod 2). Therefore, these q_n are, in any case, *odd* and we have

$$\sigma^2 \equiv \sum_n \frac{q_n^2}{(2n)^{2\mu}} \equiv \sum_n \frac{1}{(2n)^{2\mu}} \pmod{2} .$$

But this contributes (mod 2) to x^k, 0 if k is odd and 1 if k is even. The coefficient of x^{k+1} in (60) is therefore $\equiv -1 - 2 \equiv (-1)^k \pmod{4}$, if k is even, and $\equiv (-1) = (-1)^k \pmod{4}$ if k is odd. This proves (47).

To prove (48), we extract from the series (42) the part with the denominators which are powers of 2 and denote it by

$$\sigma^* = \sum_{\varkappa=1}^{\infty} \frac{g_{\varkappa}}{2^{\varkappa\mu}} . \qquad (61)$$

Then we have, obviously, $g_k = (-1)^{k-1} \pmod{4}$. We keep now in (59) on both sides only the terms which have in the denominators pure powers of 2 and obtain

$$\sigma^* = \frac{x - 2x^2}{1 - x} - 2\,\frac{x - 8x^2}{1 - 4x}\,\sigma^{*2} - \sum_{\varkappa-1}^{\infty} \frac{T_{2^{\varkappa}}(\sigma^*)}{2^{(\varkappa+1)\mu}} ,$$

since $T_2(\sigma)$ identically vanishes. Taking this modulo 8, we have by (51°)

$$\sigma^* \equiv x - x^2 - x^3 - \dots - 2x\sigma^{*2} \pmod{8} . \qquad (62)$$

We put here for σ^* the expression (61) and compare on both sides the coefficient of 2^{-k} (mod 8). Then we have, since we can use on the right (47), for $k > 1$:

$$g_k = -1 - 2 \sum_{\varkappa_1+\varkappa_2=k-1} g_{\varkappa_1} g_{\varkappa_2} \equiv -1 - 2 \sum_{\varkappa_1+\varkappa_2=k-1} (-1)^{\varkappa_1+\varkappa_2-2} =$$

$$= -1 - 2(k-2)(-1)^{k-1} \quad ,$$

and this proves (48), as $q_{2k} = g_{k+1}$.

§ 19. We are going now to write (59) in such a way as to make possible an easy recursive computation of the coefficients of (42). To that purpose we decompose the asymptotic expansion of σ, given by (42), and which we shall also denote by σ, in the following way, using (55),

$$\sigma = \sum_{\varkappa=1}^{\infty} \frac{1}{2^{\varkappa\mu}} \sum_{u} \frac{q_{2\varkappa-1u}}{u^{\mu}} ,$$

$$\sigma = \sum_{\varkappa=1}^{\infty} \sigma_{\varkappa} x^{\varkappa} , \tag{63}$$

$$\sigma_{\varkappa} = \sum_{u} \frac{q_{2\varkappa-1u}}{u^{\mu}} \quad (\varkappa = 1, 2, \ldots) . \tag{64}$$

§ 20. On the other hand, we have in (50) and (51):

$$R_u(\sigma) = \sum_{\varkappa=1}^{\infty} (-1)^{\varkappa} \frac{\sigma^{2\varkappa+1}}{(2\varkappa+1)!} S_{\varkappa}(u) ,$$

$$S_{\varkappa}(u) = u(u^2-1^2)(u^2-3^2)\ldots(u^2-(2\varkappa-1)^2) , \tag{65}$$

$$T_g(\sigma) = \sum_{\varkappa=2}^{\infty} (-1)^{\varkappa} \frac{\sigma^{2\varkappa}}{(2\varkappa)!} C_{\varkappa}(g) ,$$

$$C_{\varkappa}(g) = g^2(g^2-2^2)\ldots(g^2-(2\varkappa-2)^2) , \tag{66}$$

and, in particular, $T_2(\sigma) \equiv 0$. We can now write if the summation index g is running through all even integers greater than 2, using (66),

$$\sum_{\omega=2}^{\infty} (-1)^{\omega} \frac{T_{2\omega}(\sigma)}{(2\omega)^{\mu}} = \sum_{\varkappa=2}^{\infty} (-1)^{\varkappa} \frac{\sigma^{2\varkappa}}{(2\varkappa)!} \sum_{g>2}^{\infty} (-1)^{g/2} \frac{C_{\varkappa}(g)}{g^{\mu}} . \tag{67}$$

We introduce now the displacement operator H acting on μ and diminishing μ by one unit,

$$\mathrm{H}^m f(\mu) = f(\mu - m) .$$

Then we can write for the inner sum in (67)

$$\sum_{g>2} (-1)^{g/2} \frac{C_\varkappa(g)}{g^\mu} = C_\varkappa(\mathrm{H}) \sum_{g>2} \frac{(-1)^{g/2}}{g^\mu},$$

where $C_\varkappa(\mathrm{H})$ is the polynomial operator obtained from the expression $C_\varkappa(g)$ in (66) replacing there the powers of g by the corresponding powers of H. Here the sum to the right is by (56)

$$\sum_{\omega=2}^{\infty} (-1)^\omega \frac{1}{(2\omega)^\mu} = \frac{2x^2 - x}{1-x} \mathrm{U}(\mu) + x ,$$

and the inner sum in (67) becomes

$$C_\varkappa(\mathrm{H}) \frac{2x^2 - x}{1-x} \mathrm{U}(\mu) + C_\varkappa(\mathrm{H})\, x .$$

We have, however,

$$\mathrm{H}x = 2x , \quad \mathrm{H}^2 x = 4x , \quad (\mathrm{H}^2 - 2^2)\, x = 0$$

and it follows from the expression of $C_\varkappa(g)$ in (66) that $C_\varkappa(\mathrm{H})\, x = 0$ for $\varkappa \geq 2$. We obtain, therefore, finally

$$\sum_{\omega=2}^{\infty} (-1)^\omega \frac{\mathrm{T}_{2\omega}(\sigma)}{(2\omega)^\mu} = \sum_{\varkappa=2}^{\infty} (-1)^\varkappa \frac{\sigma^{2\varkappa}}{(2\varkappa)!} C_\varkappa(\mathrm{H}) \frac{2x^2 - x}{1-x} \mathrm{U}(\mu) . \qquad (68)$$

§ 21. In the same way, we obtain from (65)

$$\sum_{\omega=1}^{\infty} (-1)^\omega \frac{\mathrm{R}_{2\omega+1}(\sigma)}{(2\omega+1)^\mu} = \sum_{\varkappa=1}^{\infty} (-1)^\varkappa \frac{\sigma^{2\varkappa+1}}{(2\varkappa+1)!} \sum_{u>1} (-1)^{\frac{u-1}{2}} \frac{S_\varkappa(u)}{u^\mu} .$$

Here the inner sum is

$$S_\varkappa(\mathrm{H}) \sum_{u>1} \frac{(-1)^{\frac{u-1}{2}}}{u^\mu} = S_\varkappa(\mathrm{H})\, (\mathrm{L}(\mu) - 1) = S_\varkappa(\mathrm{H})\, \mathrm{L}(\mu) ,$$

since $S_\varkappa(H)\,1 = S_\varkappa(1) = 0$ for $\varkappa \geqq 1$. We obtain

$$\sum_{\omega=1}^{\infty} (-1)^\omega \frac{R_{2\omega+1}(\sigma)}{(2\omega+1)^\mu} = \sum_{\varkappa=1}^{\infty} (-1)^\varkappa \frac{\sigma^{2\varkappa+1}}{(2\varkappa+1)!} S_\varkappa(H)\,L(\mu)\,. \tag{69}$$

With (68) and (69), (59) becomes

$$L(\mu-1)\,\sigma = \frac{x-2x^2}{1-x} U(\mu) - 2\sigma^2 \frac{x-8x^2}{1-4x} U(\mu-2) - \tag{70}$$

$$-\sum_{\varkappa=2}^{\infty} (-1)^\varkappa \frac{\sigma^{2\varkappa}}{(2\varkappa)!} C_\varkappa(H) \left(\frac{2x^2-x}{1-x} U(\mu)\right) - \sum_{\varkappa=1}^{\infty} (-1)^\varkappa \frac{\sigma^{2\varkappa+1}}{(2\varkappa+1)!} S_\varkappa(H)\,L(\mu)\,.$$

§ 22. We introduce now in (70) for σ the expression (63), develop all powers of σ and compare the coefficient of each power of x on the left and on the right. Observing that the 2nd, 3rd, and 4th terms on the right begin respectively with x^3, x^4 and x^3, we obtain then for σ_1 and σ_2

$$L(\mu-1)\,\sigma_1 = U(\mu)\,, \quad L(\mu-1)\,\sigma_2 = -U(\mu)\,,$$

$$\sigma_1 = -\sigma_2 = \frac{U(\mu)}{L(\mu-1)}\,. \tag{71}$$

It follows then from (53) and (54), p running trough all odd primes,

$$-\sigma_2 = \sigma_1 = \prod_{p>2} \left[\left(1 - \frac{(-1)^{\frac{p-1}{2}} p}{p^\mu}\right)\left(1 + \frac{1}{p^\mu} + \frac{1}{p^{2\mu}} + \dots\right)\right] =$$

$$= \prod_{p>2} \left[1 + \left(1 - (-1)^{\frac{p-1}{2}} p\right)\left(\frac{1}{p^\mu} + \frac{1}{p^{2\mu}} + \frac{1}{p^{3\mu}} + \dots\right)\right]\,. \tag{72}$$

We obtain then as the numerator corresponding to $(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k})^\mu$ the expression

$$\prod_{\varkappa=1} \left(1 - (-1)^{\frac{p_\varkappa-1}{2}} p_\varkappa\right) = \prod_{\varkappa=1} (1 \pm p_\varkappa)$$

where in each factor the plus or minus sign is to be taken in such a way that this factor is divisible by 4. We obtain, therefore, for the numerators in σ_1 and σ_2

$$q_u = - q_{2u} = \prod_{p/u} (1 \pm p) \, . \tag{73}$$

Comparing, on both sides, the coefficients of x^3 we have, since $S_1(u) = u^3 - u$,

$$L(\mu - 1)\,\sigma_3 = - U(\mu) - 2\sigma_1^2\, U(\mu - 2) + \tag{74}$$
$$+ \frac{\sigma_1^3}{6}\,(L(\mu - 3) - L(\mu - 1)) \, .$$

From this formula, we can again express σ_3 by means of the functions $U(s)$, $L(s)$ and proceeding in the same way obtain for a general $\sigma_\varkappa$, interpreted as a *formal* Dirichlet series, expressions containing only $\sigma_1, \ldots, \sigma_{\varkappa-1}$. However, already the expression foı σ_3 becomes essentially more complicated than those of σ_1 and σ_2. We give here only the expression for the coefficient of $\frac{1}{p^\mu}$ for an odd prime number p in σ_3:

$$\frac{1}{6}\left((-1)^{\frac{p-1}{2}} p - 1\right)\left((-1)^{\frac{p-2}{2}} p - 5\right)\left((-1)^{\frac{p-1}{2}} p - 6\right) \, ,$$

which is easily obtained from (74) and has been derived directly by Dr. J. C. P. Miller. It is easy to see that this is always divisible by 16.

BIBLIOGRAPHY

1. D. H. Lehmer, On the Maxima and Minima of Bernoulli Polynomials. *Am. Math. Monthly*, *47* (1940), 533-538.
2. N. E. Nörlund, Mémoire sur les Polynômes de Bernoulli. *Acta Math.*, *43* (1920), 121-196.

Mathematics Research Center of the U.S. Army,
Madison, Wisconsin.