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states: If n is not a power of 2, and $3 \leqq h \leqq \infty$, then $F(Z_2, n)^h$ cannot be realized.

Subsequent revelations showed that the situation is even worse: the preceding examples of realizations of F's are nearly all that exist. The method for proving this uses the fact that the cyclic reduced p^{th} powers, which operate in the algebra $H^*(X; Z_p)$, satisfy certain relations. In the next three sections we will discuss these operations, and their implications for the realization problem.

3. Construction of the squaring operations.

Before presenting the algebra of the reduced power operations, it may be worthwhile to give a recently improved form of the definition of the operations themselves. For simplicity we restrict ourselves to the case of the prime 2.

Let π be a cyclic group of order 2 with generator T ($T^2 = 1$). Let W be an acyclic complex on which π acts freely. Algebraically, W is a free resolution of Z over π. Geometrically, W can be taken to be the union of an infinite sequence of spheres $S^0 \subset S^1 \subset \ldots \subset S^n \subset \ldots$ where each is the equator of its successor, and T is the antipodal transformation. Identifying equivalent points under π gives the infinite dimensional real projective space P with W as its 2-fold covering. Recall that $H^*(P; Z_2)$ is the polynomial ring $F(Z_2, 1)^\infty$ on the one dimensional generator U.

Let K be any space, form the cartesian product $W \times K \times K$, and let π act in this space by $T(w, x, y) = (Tw, y, x)$. Then T has no fixed points. Identifying equivalent points gives a space, denoted by $W \times_\pi K^2$, which is covered twice by $W \times K^2$. Imbed $W \times K$ in $W \times K^2$ by $(w, x) \rightarrow (w, x, x)$. Then π transforms $W \times K$ into itself with $T(w, x) = (Tw, x)$. It follows that $W \times K$ covers a subspace $P \times K$ imbedded in $W \times_\pi K^2$. This gives the diagram

$$(3.1) \qquad P \times K \xrightarrow{i} W \times_\pi K^2 \xleftarrow{h} W \times K^2 \xrightarrow{g} K^2$$

where i is the inclusion, h is the covering, and g is the obvious projection.

The squaring operations will be defined by the following diagram of cohomology with coefficients Z_2:

$$(3.2) \qquad H^q(K) \xrightarrow{\Psi} H^{2q}(W \times_\pi K^2) \xrightarrow{i^*} H^{2q}(P \times K) = \sum_{j=0}^{2q} H^{2q-j}(P) \otimes H^j(K) .$$

The function Ψ is still to be defined. The equality on the right is the standard decomposition of the Künneth theorem $H^*(P \times K) = H^*(P) \otimes H^*(K)$.

To define Ψ, recall that h defines an isomorphism

$$H^*(W \times_\pi K^2) \approx H^*_\pi(W \times K^2)$$

where H^*_π denotes the cohomology of $W \times K^2$ based on cochains which are invariant under the action of π. This isomorphism was studied first by Eilenberg [10] who called it the *equivariant* cohomology. To define Ψ, it suffices therefore to define $\Psi'\colon H^q(K) \to H^{2q}_\pi(W \times K^2)$. For simplicity, assume K is a cell complex, and that $W \times K^2$ has as cells the products of cells of its factors. Then g in 3.1 is a cellular mapping. Let u_1 be a q-cocycle representing $u \in H^q(K)$. Then $u_1 \otimes u_1$ is an invariant $2q$-cocycle of $K \times K$ where π acts by $T(x, y) = (y, x)$. Since g is cellular, it induces a cochain mapping $g^\#$. Since $gT = Tg$, it follows that $g^\#(u_1 \otimes u_1)$ is an invariant cocycle, and it thereby represents an element $\Psi' u$ in $H^{2q}_\pi(W \times K^2)$. The fact that the π-cohomology class of $g^\#(u_1 \otimes u_1)$ depends only on the class of u_1 can be proved using Lemma 5.2 in [19]. This completes the definition of Ψ' and hence of Ψ.

If $x \in \mathrm{H}^q(K)$, by 3.2 the composition $i^* \Psi x$ decomposes into a sum. Since $H^*(P)$ is the polynomial ring in U, this sum has the form $\Sigma_j U^{2q-j} \otimes v_j$ where $v_j \in H^j(K)$ is a uniquely defined function of x. It can be shown that, for $j < q$, each $v_j = 0$. The remaining v_j are called the *reduced squares* of x. Thus

$$(3.3) \qquad i^* \Psi x = \sum_{i=0}^{q} U^{q-i} \otimes \mathrm{Sq}^i x .$$

The advantage of this definition is that it analyzes the previous definition in terms of two standard operations (the i^*

and the Künneth formula) and the single new operation Ψ. This simplifies the derivation of the properties of the Sq^i, and illuminates their origin.

Note that the projection $W \times_\pi K^2 \to P$ is a fibration with fibre K^2. For each $x \in H^q(K)$, $x \otimes x$ is a cohomology class of the fibre. The element Ψx is a canonical extension of $x \otimes x$ to a class on the total space.

4. The algebras of reduced power operations.

The definition of the reduced powers, given above for complexes, extends to the Čech cohomology of general spaces by taking direct limits of the operations in the nerves of coverings. The extension to the singular theory, by the method of acyclic models, has been carried through by Araki [4].

The main property of the squares is that

$$\mathrm{Sq}^i \colon H^q(X; Z_2) \to H^{q+i}(X; Z_2)$$

is a homomorphism for each space X and each $i \geqq 0$, and if $f\colon X \to Y$ is a mapping, Sq^i commutes with the induced homomorphism f^* of cohomology. The principal algebraic properties are

(4.1) $$\mathrm{Sq}^0 = \text{identity}.$$

(4.2) $\mathrm{Sq}^1 =$ the Bockstein operator β of the coefficient sequence

$$0 \to Z_2 \to Z_4 \to Z_2 \to 0\,.$$

(4.3) If $\dim x = n$, then $\mathrm{Sq}^n x = x^2$.

(4.4) If $\dim x = n$, then $\mathrm{Sq}^i x = 0$ for all $i > n$.

(4.5) The Adem relations [2]: If $a < 2b$, then

$$\mathrm{Sq}^a\,\mathrm{Sq}^b = \binom{b-1}{a}\mathrm{Sq}^{a+b} + \sum_{j=1}^{[a/2]}\binom{b-j-1}{a-2j}\mathrm{Sq}^{a+b-j}\,\mathrm{Sq}^j\,.$$

(4.6) The Cartan formula [6]: If $x, y \in H^*(X; Z_2)$, then

$$\mathrm{Sq}^i(xy) = \sum_{j=0}^{i}(\mathrm{Sq}^j x)(\mathrm{Sq}^{i-j} y)\,.$$