Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 8 (1962)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SOME NOTES ON BRITISH CALCULUS TEXT BOOKS 1900/1960 .

Autor: Cockcroft, W. H.

DOI: https://doi.org/10.5169/seals-37960

SOME NOTES ON BRITISH CALCULUS TEXT BOOKS 1900/1960[1]

by W. H. Cockcroft

The educational pattern in Great Britain is set essentially by the examination system which we have. Any study of the teaching of a subject in our schools must therefore contain an examination of the texts in use in the schools, for, by interpretation, they mirror the examination syllabi and in this sense reflect the teaching. In addition they are of course one of the main supports of the new teacher when faced with a class. As such they should surely be written with adequate information in mind about the people who will read them. They should not be just minor treatises nor emasculated versions of more formal texts, but should by some means or other express ways of communicating intuitive ideas.

The pre-1914 texts in " the calculus " which one finds, are often later editions of texts first published around 1895. Let me pick out two for brief mention. Both claim to be introductory and certainly both were in use in the 1930's, in schools and universities. In the latter, of course, texts not aimed at the schoolboy inevitably affected the teaching of those who used them in their training.

The first of the two commences with an examination of the notions of sequences and their limits, and of series. Definitions are quite precise and are led into by concrete examples. There are no false statements. It is perhaps unfortunate that the assumption that a monotonic increasing sequence, which is bounded above, must have a limit, is stated in the middle of a long paragraph on the subject and is not isolated or made conspicuous in any way. But this was a mode of writing of the period, and at least the author realizes that certain basic assumptions are necessary in an introductory calculus course, which is naturally not going to begin with a rigorous analysis of the

[1]) Talk delivered at the Lausanne Colloquium, organised by I.C.M.I., june 1961.

structure of the real numbers. In introducing continuous functions, again there is precision of definition, but in general theory the ideas begin to get mixed up, and there are sections which the uninitiated might think were giving formal proofs, or might easily be expanded so as to be precise. In fact a fair degree of sophistication would be needed to put things in order. Once past this stage the book produces the usual technique of introductory calculus; integration, differential equations, etc., and finally Taylor's series, are all dealt with. It is not too difficult to continue to be critical. But at least one might imagine, over a period of time, that out of such a book could come a good introductory text.

One cannot feel this about the second text which describes itself as " a sound introduction to a study of the differential calculus, suitable for a beginner." Here right from the start there are many false statements. The limiting process is introduced in such a way that the student is tempted to assume all functions continuous, for the author speaks of the limit of $f(x)$ when $x = a$. Or again, the differentiation of x^n is introduced by using the binomial expansion of $(a+b)^n$, whatever the value of n, rational or not.

It is far too easy to continue being highly critical of this book. It is perhaps more reasonable to ask whether the treatment went unchallenged. Let me simply observe that when the author allowed an expanded version of his work, written in a similar fashion, to be republished later, a reviewer observed that the author appeared to have been asleep for forty years and had not realized the work which had been done in analysis during his lifetime. This did not stop the successful commercial publication of the book; publication of school texts in Great Britain is a purely commercial matter. No *direct* guidance is given by any Ministry to the teachers.

It is difficult to believe that the student who worked from typical school texts of this period could really gain much more than a facility in the techniques of the calculus. Let me say again that I do not suggest that an introductory course in analysis must begin with a rigorous study of the real numbers, but I do suggest that, even in the most elementary text-book, the professional mathematician should find evidence which suggests that the author is aware of the problems and difficulties in the subject when it is treated in a rigorous fashion; and that furthermore the author should show a high degree of skill in the use of intuitive thinking, particularly in the early stages, so that the student is led naturally to think on correct lines.

One might of course argue that it was too early in the later part of the last century to expect the ideas of Weierstrass on the foundations of analysis to have been so familiar as to shine through clearly in an elementary treatment. However if one turns to the reviews and criticisms of texts in this pre-first world war period, one finds much to contradict this. The reports prepared in Great Britain (and published in 1912), for the International Commission on the teaching of mathematics, contain much general criticism. University Professors claim in these that " the general level of books and teaching is little better than it used to be." One Professor does not want "to drive every mediocre school boy crazy with δ and ε". He understands that there is a stage in everybody's education when perfect rigour is a quite unpracticable ideal but he does ask that school boys should not have their minds filled with the pitiable nonsense that he himself was taught at school and, in particular, that when it is clearly impossible for reasonably accurate proofs to be given, then the pretence of proof should be abandoned. For the best children he asks that they should have decent reasoning put before them right from the beginning. He makes the reasonable request, for example, that the differentiation of x^n should be confined in the first instance to the case when n is rational (as was done in the better of the texts I mentioned in detail above, but as was not done in the other). Of course he observes that a proof using the Binomial Series is just what one should not give. He closes by observing that he has able students who " will never really understand analysis because they will never recover from the poison administered to them early in life ", and he observes that better methods of teaching are surely practicable in the light of the example of France and Germany.

This then from a Professor. It is impossible to leave his comments without telling you of his remarks on a famous calculus text of the period which was subtitled " what one fool can do, other fools can do also ". It is described by him in most caustic terms as " a thoughtless adoption of bad traditional methods, with perverse (though traditional) ingenuity, with which it makes difficult and obscure what ought to be easy and illuminating."

But what of the school teachers of the period? Here is one writing in the same volume of reports. He complains that many authors appear to think that the meaning of difficult words can be revealed merely by the use of italics and that there is lack of appreciation of the need for assigning definite meanings to words used in a technical sense. He gives many quotations, observing for example that authors confuse the idea of a function with that of the value of a function; that the notion of a limit is explained by appealing to a " principle of continuity "—putting the cart before the horse; or again that, by implication, texts leave the reader with the idea that the limit of a function is necessarily a value of the function. On the subject of infinite series he observes that nearly every writer spoils his arguments concerning elementary tests of convergence by tacitly assuming that the series in question converges. He offers honest arguments leading from concrete examples, through intuitive ideas, to the precise definitions of limits of functions of a real variable, which he would prefer to see used, together with exercises to help the student in this process. Let me remind you that this is a *school master* of the period who is writing.

If one turns now to the period between the wars one easily finds texts designed to meet the requirements of the examination system of the time. Probably the most popular of these makes no pretence whatsoever to discuss any limiting process other than that of the chord of a curve approaching the tangent at one end of the chord. Differentials in their most naive geometric sense are introduced early in the book. One can only presume, I think, that this is done to justify in some vague sense the notation $\frac{dy}{dx}$ which seems always to be preferred to the functional notation $f'(x)$. Why, going further, second and higher order differentials are introduced, I find impossible to decide. Again, by successive differentiation of a power series, Maclaurin's Theorem is stated, with the observation of what a general form of the remainder term would be, but with the statement that it is beyond the scope of the book to prove the formula for this remainder. One could surely ask that in addition examples be given, say of log $(1+x)$, or $\tan^{-1}x$, in which the remainder term can be calculated without recourse to Rolle's Theorem. To be sure, such a book has little of the verbosity of the older books, but competence and accuracy in dealing with large numbers of exercises seems always the aim. In fact, do the difficulties of preparing children for large scale external examinations outweigh the insights and feelings of good teachers, and cause such books to be so popularly used?

Another text much in vogue at one time between the wars claimed in its preface that " appeal would be made throughout the course to the method of ' pictures and plausibility ' ". The authors hoped that the student would have nothing to unlearn should he elect afterwards to proceed to a rigorous course of modern analysis and that he would see throughout the book what is definitely proved and what is merely allowed to rest on a basis of geometric imagination. These ideals are I think to be applauded, but in practice reading the book one wonders just how obvious it is what is proof and what rests on intuition. In an introductory course on a subject, dealt with in this way, surely the student has a right to be explicitly told when no proof is given and only an informal discussion has taken place. These discussions should moreover not be phrased as mathematical arguments; I am convinced that when no proof is intended, the greatest care should be taken to avoid the typical forms of words used in arguments giving strict proof. Here I feel the book let the student down. These clear and explicit observations of what is proof, and what is not, are missing, and intuitive observations read, to the uninitiated, like " proofs ".

If one turns to reviews, again one gets criticism.In the type of book which gives no idea of honest limiting processes one finds criticised the notion that it is impossible to introduce the idea of a limit to children, without violating logic. Unfortunately one finds together with observations of the defects of a book in which the idea of a limit is neither explained nor defined, the suggestion that an experienced teacher would know how to counteract the defects. This I feel is a pernicious doctrine but one which one knows to be all too common. In particular, it ignores completely the problems of the young teacher trying to develop his courses and requiring all the help he can get.

Of general criticism of the state of mathematical teaching between the wars, there is a great deal to be found in the Mathematical Gazettes of the time. One finds articles of many kinds, both general and particular. Thus a mathematically trained Professor of Education, concerned with ways in which pedagogy could be improved in the school in the light of the ways in which mathematical thinking had moved at all levels since the beginning of the 19th century, draws attention to the many articles and lectures given by Klein on the concept of function and the need for consideration of how the basic ideas involved therein could be introduced into school texts. Or again there is a whole series of articles, including one by Hadamard, on the need for abandoning, in teaching, the complicated notions of differentials. There are to be found many detailed discussions by school teachers of the need for logical thinking at a high level on the part of authors of elementary calculus books. There are constant references to bad proofs and illogical arguments in existing books, together with many positive suggestions for improvement.

There is little need to continue in this way. I hope it is clear what kind of picture one forms. The text books had indeed changed and reflected detailed criticism, but still teachers in both schools and universities were dissatisfied and said so, in many cases in forceful terms.

What then has happened since the last war? One thing certainly, there have been attempts to teach calculus even earlier in our schools—some teachers would say with even less guidance. Once again the teaching at this level has become caught up in the external examination system and more often than not it would appear that the student seeing a question on the calculus involving x^3 gives an automatic response $3x^2$, not really backed up by understanding.

Once again, and very briefly, many texts have been published in which the process of avoiding wrong detail has continued. The better ones have tended to be expensive, and to have naturally required reading with care. Here is one difficulty; a tradition has grown up (because of the texts and the examination system?) among our boys and girls, of reading texts mainly to find exercises to improve their technique but not necessarily their understanding. In consequence the more expensive book, needing to be read with care, is not popular.

One series of texts has certainly proved popular in first year analysis courses for engineers in Universities. These have been produced by University lecturers at, or connected with, Manchester University, under the editorship of W. Ledermann. These are cheap, brief, and honest, and I think might point the way to the kind of text many of us would like to see used, at least by the teacher in his preparation of his lessons. But there seems little reason why in some form they could not be the basis of good texts for the children. Certainly successful *first* courses on the calculus for students with no previous experience have for many years been given in the Scottish Universities following the pattern of these books.

The complaints from the Universities which were made in 1912 are still being heard, and I know from my own experience the difficulties which one faces with average students in their first term at University, when they have apparently been drilled in technique and see no virtue in analysing the ideas which justify this technique. One has to search hard for simple exercises in which they find their technique leads them astray, so that their resistance to the processes of analysis is weakened. Personally, I see no reason to never have *some* resistance to break down before beginning the process of re-creation, but I do expect that this last process should be far more easily linked with what has gone before than is the case at present. One wonders why the students in mathematics were attracted to the subject in the University. Is it just because they were better at "sums" than others at school? Do they have *any* idea of what the subject is like at University, or why it should be like this? Is it just an examination subject where they can pick up high marks?

It is surely not too much to ask that the text books which we have in our schools should give the student who wishes to take up mathematics a sufficient understanding of the nature of the subject. Recent conferences, and the work of our Mathematical Association in preparing reports on the teaching of what they considered should be the content of school courses at various levels, lead one to believe that their is a genuine desire to bridge the gap between school and University. On both sides it is felt more strongly than ever that the teacher should be given every help to teach the introductory stages of a subject in the way which reflects the road ahead in the subject.

One would hope to see in the next few years more development and much expansion of the present means of collaboration between mathematicians, so that the divergence of outlook between school and University as mirrored in our texts particularly, could be narrowed. I have already observed that University lecturers have produced books which one could easily imagine being used as a basis for discussion of what constitute acceptable texts. But it must surely be as a basis only, for we need more than an accurate text; we must understand how children form intuitive concepts, and how these can be built up into mathematical concepts. The insight of both the University teacher and the school teacher from their respective points of view are needed. We do not want mere mathematical pedantry, but we do want texts with their sights on the road ahead. A vigorous programme of co-operation is surely called for, so that after over 50 years of diagnosis of the problem, of criticism and of argument, we shall at last see an honest, positive, effort to cure the patient. Above all, our teachers deserve texts and commentaries thereon which will help the experienced and inexperienced alike to see on the one hand the ways in which intuitive understanding can be created where it is needed, and, on the other hand, the ways in which logical understanding can be created where it is appropriate and to be expected. We must have co-operation to be more sure of what boys and girls really can do and think about; we are surely failing in our duty to them if, as teachers, we do not make this effort.

University of Southampton
England.