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Artikel: ON THE CONSTRUCTION OF RELATED EQUATIONS FOR THE ASYMPTOTIC THEORY OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS ABOUT A TURNING POINT

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one, and accordingly admits of an analytic solution for $(a^{(0)})_0$ provided the matrix multiplier of this vector on the left is non-singular. This condition is assured by the relation (3. 4).

Now we may proceed by induction. Assuming that the vectors $(a^{(0)})_j$ for $j = 1, 2, \ldots, (\nu-1)$, have been determined and are analytic, the right-hand member of the equation (7. 7) is known. As in the case $\nu = 0$, so now, the equation is analytically solvable. The solutions for the successive values $\nu = 0, 1, 2, \ldots, (r-1)$, yield the coefficients (6. 7) for which the functions $\eta_i(z, \lambda)$, as given by the formulas (6. 8), fulfill the relations (6. 5).

8. On linear independence.

With the functions $a_j^{(0)}(z, \lambda)$ now at hand, we have at our disposal the n known functions $y_j(z, \lambda)$, $j = 1, 2, \ldots, q$, which are the solutions of the differential equation (6. 3), and $\eta_i(z, \lambda)$, $i = 1, 2, \ldots, p$, which are given by the formulas (6. 8). We shall show that these functions are linearly independent.

Let the Wronskians of the entire set and of the respective sub-sets be denoted respectively by W_n, $W_q(y)$ and $W_p(\eta)$. If the usual form

$$W_n = \begin{vmatrix} y_1 & - & - & - & y_q & \eta_1 & - & - & - & \eta_p \\ Dy_1 & - & - & - & Dy_q & D\eta_1 & - & - & - & D\eta_q \\ - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - \\ D^{n-1}y_1 & - & - & - & D^{n-1}y_q & D^{n-1}\eta_1 & - & - & - & D^{n-1}\eta_p \end{vmatrix} \quad (8.\ 1)$$

is modified by adding to each of the last p rows suitable multiples of the preceding ones, the formula can be made to appear thus

$$= \begin{vmatrix} y_1 & - & - & - & - & y_q & \eta_1 & - & - & - & \eta_p \\ Dy_1 & - & - & - & - & Dy_q & D\eta_1 & - & - & - & D\eta_p \\ - & - & - & - & - & - & - & - & - & - & - \\ D^{q-1}y_1 & - & - & - & - & D^{q-1}y_q & D^{q-1}\eta_1 & - & - & - & D^{q-1}\eta_p \\ m^*(y_1) & - & - & - & - & m^*(y_q) & m^*(\eta_1) & - & - & - & m^*(\eta_p) \\ Dm^*(y_1) & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - \\ D^{p-1}m^*(y_1) & - & - & - & - & D^{p-1}m^*(y_q) & D^{p-1}m^*(\eta_1) & - & - & - & D^{p-1}m^*(\eta_p) \end{vmatrix} \quad (8.\ 2)$$

In this, however, each of the elements occupying a position in one of the first q columns and in one of the last p rows is zero. The formula therefore reduces at once to

$$W_n = W_q(y)\,T, \tag{8.3}$$

with

$$T = \begin{bmatrix} m^*(\eta_1) & - & - & - & - & m^*(\eta_p) \\ Dm^*(\eta_1) & - & - & - & - & Dm^*(\eta_p) \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ D^{p-1}m^*(\eta_1) & - & - & - & - & D^{p-1}m^*(\eta_p) \end{bmatrix}. \tag{8.4}$$

Now $m^*(\eta_j)$ is given by the formula (6. 15). If this is repeatedly differentiated, and at each step the element $D^p v_j$ is eliminated by use of the equation (6. 1), the results are the formulas

$$D^i m^*(\eta_j) = \lambda^q D^i v_j + \lambda^{q+i-r} \sum_{\mu=0}^{p} \lambda^{1-\mu} \sigma_{\mu,r}^{(i)} D^{\mu-1} v_j, \quad i = 0, 1, 2, \dots . \tag{8.5}$$

We may write this also, with the use of the symbol $\delta_{i,j}$ to denote 1 when $j = i$ and 0 when $j \neq i$, in the form

$$D^{i-1} m^*(\eta_j) = \lambda^{q+i-l} \sum_{\mu=1}^{p} \left\{ \delta_{i,\mu} + \frac{\sigma_{\mu,r}^{(i-1)}}{\lambda^r} \right\} \frac{D^{\mu-1} v_j}{\lambda^{\mu-1}}. \tag{8.6}$$

This shows, now, at once, that the determinant T can be factored, thus

$$T = \lambda^{pq} E W_p(v) \tag{8.7}$$

in which E is the determinant whose element in the i^{th} row and j^{th} column is indicated thus

$$E = \left| \delta_{i,j} + \frac{\sigma_{j,r}^{(i-1)}}{\lambda^r} \right|. \tag{8.8}$$

It is clear that E differs from 1 by terms of at least the degree r in $1/\lambda$. Since $W_p(v)$ and $W_q(y)$ are non-vanishing, it follows from (8. 3) and (8. 7) that the same is true of W_n.

9. The related equation.

We are prepared now to make the construction toward which this entire discussion has been directed.

Consider the equation

$$L^*(u) = 0 . \tag{9.1}$$

with

$$L^*(u) = \frac{1}{T}\begin{bmatrix} m^*(\eta_1) & - - - - - - & m^*(\eta_p) & m^*(u) \\ Dm^*(\eta_1) & - - - - - - & Dm^*(\eta_p) & Dm^*(u) \\ - & - - - - - - & - & - \\ - & - - - - - - & - & - \\ D^{p-1}m^*(\eta_1) & - - - - - & D^{p-1}m^*(\eta_p) & D^{p-1}m^*(u) \\ l^*(m^*(\eta_1)) & - - - - - & l^*(m^*(\eta_p)) & l^*(m^*(u)) \end{bmatrix} \cdot \tag{9.2}$$

T being the determinant given in (8. 4). This is clearly a differential equation of the n^{th} order in u, for which each one of the functions $y_j(z, \lambda)$ and $\eta_i(z, \lambda)$ is a solution. For if η_i is substituted for u two of the columns of the determinant (9. 2) are the same, and if u is replaced y_j every element of the last column vanishes. Because the n solutions thus produced are linearly independent the solutions of the equation (9. 1) are completely known.

The co-factor of the element $l^*(m(u))$ in the formula (9. 2) is the determinant T. The expansion of the formula thus gives it the aspect

$$L^*(u) = l^*(m^*(u)) - \sum_{\nu=1}^{p} \frac{T_\nu}{T} D^{p-\nu} m^*(u) , \tag{9.3}$$

where T_v is the determinant that is obtainable from the formula (8. 4) by replacing its elements $D^{p-\nu} m^*(\eta_j)$ by $l^*(m^*(\eta_j))$.

From the formula (8. 5) it is seen that

$$l^*(m^*(\eta_j)) = \lambda^n \sum_{\nu=1}^{p} \frac{\tau_\nu(z, \lambda)}{\lambda^r} \cdot \frac{D^{\mu-1} v_j}{\lambda^{\mu-1}} \tag{9.4}$$

with

$$\tau_\nu(z, \lambda) = \sum_{k=0}^{p} \bar{\beta}_k(z, \lambda)\, \sigma_{\nu, r}^{(p-k)}(z, \lambda) . \tag{9.5}$$

The replacements which change T to T_ν are thus seen to be ones which replace

$$\lambda^{n-\nu}\left\{\delta_{p-\nu,j}+\frac{\sigma_{j,r}^{(p-\nu)}}{\lambda^r}\right\} \quad \text{by} \quad \lambda^n\frac{\tau_\nu}{\lambda^r}.$$

It follows that

$$\frac{T_\nu}{T}=\lambda^\nu\frac{\theta_\nu(z,\lambda)}{\lambda^r},$$

with some function $\theta_\nu(z, \lambda)$ which is bounded over the z and λ domains. This gives to the relation (9. 3) the form

$$L^*(u) = l^*(m^*(u))-\frac{1}{\lambda^r}\sum_{\nu=1}^{p}\lambda^\nu\theta_\nu D^{p-\nu}m^*(u). \quad (9.7)$$

With the substitution of the expression for $D^{p-\nu}m^*(u)$, as it may be obtained from (4. 3) by writing $\bar{\gamma}_{i-s}$ in the place of γ_{i-s}, it is found that

$$L^*(u) = l^*(m^*(u))-\frac{1}{\lambda^r}\sum_{j=1}^{n}\lambda^j\omega_j(z,\lambda)D^{n-j}u, \quad (9.8)$$

with

$$\omega_j(z,\lambda) = \sum_{\nu=1}^{p}\sum_{s=0}^{p}\lambda^{-s}\binom{p-\nu}{s}\theta_\nu D^s\bar{\gamma}_{\mu-\nu-s}.$$

A comparison of this with the earlier result (6. 6) shows that

$$L^*(u) = L(u)-\frac{1}{\lambda^r}\sum_{j=1}^{n}\lambda^j\{\varepsilon_j(z,\lambda)+\omega_j(z,\lambda)\}D^{n-j}u. \quad (9.9)$$

The equation (9. 1), whose solutions are completely known, thus has coefficients which differ from those of the given equation (2. 1) only by terms that are of at least the r^{th} degree in $1/\lambda$. It is, therefore, by definition, a related equation.

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