

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 8 (1962)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE CONSTRUCTION OF RELATED EQUATIONS FOR THE ASYMPTOTIC THEORY OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS ABOUT A TURNING POINT
Autor: Langer, Rudolph E.
Kapitel: 9. The related équation.
DOI: <https://doi.org/10.5169/seals-37963>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 18.04.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

9. THE RELATED EQUATION.

We are prepared now to make the construction toward which this entire discussion has been directed.

Consider the equation

$$L^*(u) = 0. \quad (9.1)$$

with

$$L^*(u) = \frac{1}{T} \begin{bmatrix} m^*(\eta_1) & - & - & - & - & m^*(\eta_p) & m^*(u) \\ Dm^*(\eta_1) & - & - & - & - & Dm^*(\eta_p) & Dm^*(u) \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ D^{p-1}m^*(\eta_1) & - & - & - & - & D^{p-1}m^*(\eta_p) & D^{p-1}m^*(u) \\ l^*(m^*(\eta_1)) & - & - & - & - & l^*(m^*(\eta_p)) & l^*(m^*(u)) \end{bmatrix}. \quad (9.2)$$

T being the determinant given in (8.4). This is clearly a differential equation of the n^{th} order in u , for which each one of the functions $y_j(z, \lambda)$ and $\eta_i(z, \lambda)$ is a solution. For if η_i is substituted for u two of the columns of the determinant (9.2) are the same, and if u is replaced y_j every element of the last column vanishes. Because the n solutions thus produced are linearly independent the solutions of the equation (9.1) are completely known.

The co-factor of the element $l^*(m(u))$ in the formula (9.2) is the determinant T . The expansion of the formula thus gives it the aspect

$$L^*(u) = l^*(m^*(u)) - \sum_{v=1}^p \frac{T_v}{T} D^{p-v} m^*(u), \quad (9.3)$$

where T_v is the determinant that is obtainable from the formula (8.4) by replacing its elements $D^{p-v} m^*(\eta_j)$ by $l^*(m^*(\eta_j))$.

From the formula (8.5) it is seen that

$$l^*(m^*(\eta_j)) = \lambda^n \sum_{v=1}^p \frac{\tau_v(z, \lambda)}{\lambda^r} \cdot \frac{D^{\mu-1} v_j}{\lambda^{\mu-1}} \quad (9.4)$$

with

$$\tau_v(z, \lambda) = \sum_{k=0}^p \bar{\beta}_k(z, \lambda) \sigma_{v,r}^{(p-k)}(z, \lambda). \quad (9.5)$$

The replacements which change T to T_v are thus seen to be ones which replace

$$\lambda^{n-v} \left\{ \delta_{p-v, j} + \frac{\sigma_{j, r}^{(p-v)}}{\lambda^r} \right\} \quad \text{by} \quad \lambda^n \frac{\tau_v}{\lambda^r}.$$

It follows that

$$\frac{T_v}{T} = \lambda^v \frac{\theta_v(z, \lambda)}{\lambda^r},$$

with some function $\theta_v(z, \lambda)$ which is bounded over the z and λ domains. This gives to the relation (9.3) the form

$$L^*(u) = l^*(m^*(u)) - \frac{1}{\lambda^r} \sum_{v=1}^p \lambda^v \theta_v D^{p-v} m^*(u). \quad (9.7)$$

With the substitution of the expression for $D^{p-v} m^*(u)$, as it may be obtained from (4.3) by writing $\bar{\gamma}_{i-s}$ in the place of γ_{i-s} , it is found that

$$L^*(u) = l^*(m^*(u)) - \frac{1}{\lambda^r} \sum_{j=1}^n \lambda^j \omega_j(z, \lambda) D^{n-j} u, \quad (9.8)$$

with

$$\omega_j(z, \lambda) = \sum_{v=1}^p \sum_{s=0}^p \lambda^{-s} \binom{p-v}{s} \theta_v D^s \bar{\gamma}_{\mu-v-s}.$$

A comparison of this with the earlier result (6.6) shows that

$$L^*(u) = L(u) - \frac{1}{\lambda^r} \sum_{j=1}^n \lambda^j \{ \varepsilon_j(z, \lambda) + \omega_j(z, \lambda) \} D^{n-j} u. \quad (9.9)$$

The equation (9.1), whose solutions are completely known, thus has coefficients which differ from those of the given equation (2.1) only by terms that are of at least the r^{th} degree in $1/\lambda$. It is, therefore, by definition, a related equation.

REFERENCES

- [1] R. E. LANGER, The asymptotic solutions of ordinary linear differential equations of the second order with special reference to a turning point. *Trans. Amer. Math. Soc.*, vol. 67 (1949), pp. 461-490.
- [2] R. W. MCKELVEY, The solutions of second order linear ordinary differential equations about a turning point of order two. *Trans. Amer. Math. Soc.*, vol. 79 (1955), pp. 103-123.