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Artikel: ON THE CONSTRUCTION OF RELATED EQUATIONS FOR THE ASYMPTOTIC THEORY OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS ABOUT A TURNING POINT

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Kapitel: 9. The related équation.

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9. The related equation.

We are prepared now to make the construction toward which this entire discussion has been directed.

Consider the equation

$$L^*(u) = 0 . \tag{9.1}$$

with

$$L^*(u) = \frac{1}{T}\begin{bmatrix} m^*(\eta_1) & - - - - - - & m^*(\eta_p) & m^*(u) \\ Dm^*(\eta_1) & - - - - - - & Dm^*(\eta_p) & Dm^*(u) \\ - & - - - - - - & - & - \\ - & - - - - - - & - & - \\ D^{p-1}m^*(\eta_1) & - - - - - & D^{p-1}m^*(\eta_p) & D^{p-1}m^*(u) \\ l^*(m^*(\eta_1)) & - - - - - & l^*(m^*(\eta_p)) & l^*(m^*(u)) \end{bmatrix} \cdot \tag{9.2}$$

T being the determinant given in (8. 4). This is clearly a differential equation of the n^{th} order in u, for which each one of the functions $y_j(z, \lambda)$ and $\eta_i(z, \lambda)$ is a solution. For if η_i is substituted for u two of the columns of the determinant (9. 2) are the same, and if u is replaced y_j every element of the last column vanishes. Because the n solutions thus produced are linearly independent the solutions of the equation (9. 1) are completely known.

The co-factor of the element $l^*(m(u))$ in the formula (9. 2) is the determinant T. The expansion of the formula thus gives it the aspect

$$L^*(u) = l^*(m^*(u)) - \sum_{\nu=1}^{p} \frac{T_\nu}{T} D^{p-\nu} m^*(u) , \tag{9.3}$$

where T_v is the determinant that is obtainable from the formula (8. 4) by replacing its elements $D^{p-\nu} m^*(\eta_j)$ by $l^*(m^*(\eta_j))$.

From the formula (8. 5) it is seen that

$$l^*(m^*(\eta_j)) = \lambda^n \sum_{\nu=1}^{p} \frac{\tau_\nu(z, \lambda)}{\lambda^r} \cdot \frac{D^{\mu-1} v_j}{\lambda^{\mu-1}} \tag{9.4}$$

with

$$\tau_\nu(z, \lambda) = \sum_{k=0}^{p} \bar{\beta}_k(z, \lambda)\, \sigma_{\nu, r}^{(p-k)}(z, \lambda) . \tag{9.5}$$

The replacements which change T to T_ν are thus seen to be ones which replace

$$\lambda^{n-\nu}\left\{\delta_{p-\nu,j}+\frac{\sigma_{j,r}^{(p-\nu)}}{\lambda^r}\right\} \quad \text{by} \quad \lambda^n\frac{\tau_\nu}{\lambda^r}\,.$$

It follows that

$$\frac{T_\nu}{T}=\lambda^\nu\frac{\theta_\nu(z,\lambda)}{\lambda^r},$$

with some function $\theta_\nu(z,\lambda)$ which is bounded over the z and λ domains. This gives to the relation (9. 3) the form

$$L^*(u) = l^*\big(m^*(u)\big)-\frac{1}{\lambda^r}\sum_{\nu=1}^{p}\lambda^\nu\theta_\nu D^{p-\nu}m^*(u)\,. \quad (9.7)$$

With the substitution of the expression for $D^{p-\nu}m^*(u)$, as it may be obtained from (4. 3) by writing $\bar{\gamma}_{i-s}$ in the place of γ_{i-s}, it is found that

$$L^*(u) = l^*\big(m^*(u)\big)-\frac{1}{\lambda^r}\sum_{j=1}^{n}\lambda^j\omega_j(z,\lambda)D^{n-j}u\,, \quad (9.8)$$

with

$$\omega_j(z,\lambda) = \sum_{\nu=1}^{p}\sum_{s=0}^{p}\lambda^{-s}\binom{p-\nu}{s}\theta_\nu D^s\bar{\gamma}_{\mu-\nu-s}\,.$$

A comparison of this with the earlier result (6. 6) shows that

$$L^*(u) = L(u)-\frac{1}{\lambda^r}\sum_{j=1}^{n}\lambda^j\{\varepsilon_j(z,\lambda)+\omega_j(z,\lambda)\}D^{n-j}u\,. \quad (9.9)$$

The equation (9. 1), whose solutions are completely known, thus has coefficients which differ from those of the given equation (2. 1) only by terms that are of at least the r^{th} degree in $1/\lambda$. It is, therefore, by definition, a related equation.

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