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THE STONE SPACE OF A BOOLEAN RING

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This is an expository paper reproducing some of the basic results in [1] and [2].

DEFINITION 1. — *A ring B is called Boolean, if*

$$x^2 = x, \qquad \textit{for every } x \varepsilon B. \tag{1}$$

In what follows B shall represent a given Boolean ring.

The following are well known immediate consequences of Definition 1.

$$x + x = 0, \tag{2}$$

$$xy = yx, \tag{3}$$

$$xy(x+y) = 0, \tag{4}$$

for every two elements x and y of B.

NOTATION. — *In what follows, for every non-zero element x of B, $p(x)$ shall represent a prime ideal of B not containing x, i.e.,*

$x \notin p(x)$

and

$P(x)$ *shall represent the set of all prime ideals* $p(x)$, *for a given* x.

LEMMA 1. — *Let I be an ideal of B and x an element of B such that $x \notin I$. Then there exists a prime ideal $p(x)$ such that $I \subset p(x)$.*

PROOF. — By Zorn's Lemma, in view of (1) and (3), there exists a largest ideal M of B such that $I \subset M$ and $x \notin M$. It can be easily verified that the ideal M is prime [3].

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Let us observe that since 0 is an element of every ideal of B, hence, in view of Lemma 1,

$$P(x) = \phi \quad \textit{if and only if } x = 0. \tag{5}$$

Now, we prove that for every two elements x and y of B,

$$P(xy) = P(x) \cap P(y) \tag{6}$$

To prove (6), let us observe that since $p(xy)$ is an ideal not containing xy, hence, $x \notin p(xy)$ and $y \notin p(xy)$. Thus, $P(xy) \subset (P(x) \cap P(y))$. Conversely, since $p(x)$ is a prime ideal not containing x, hence, if $y \notin p(x)$ then $xy \notin p(x)$. Thus, $(P(x) \cap P(y)) \subset P(xy)$.

From (6) it follows that for every two elements x and y of B,

$$xy = x \quad \textit{implies } P(x) \subset P(y) \tag{7}$$

Since $p(x+y)$ is an ideal not containing $x+y$, hence $x \in p(x+y)$ implies $y \notin p(x+y)$. Therefore, for every two elements x and y of B,

$$P(x+y) \subset (P(x) \cup P(y)) \tag{8}$$

Furthere, in view of (4), (5) and (6),

$$P(xy) \cap P(x+y) = \phi$$

so that, in view of (8), for every two elements x and y of B,

$$P(x+y) \subset (P(x) \oplus P(y)), \tag{9}$$

where $\oplus$ is the usual set-theoretical *symmetric difference* operator. Also, let us observe that since $p(x)$ is an ideal not containing x, hence, if $p(x) \notin P(y)$ then $p(x) \in P(x+y)$. Similarly, if $p(y) \notin P(x)$ then $p(y) \in P(x+y)$. Thus,

$$P(x) \oplus P(y) \subset P(x+y)$$

so that in view of (9), for every two elements x and y of B,

$$P(x+y) = P(x) \oplus P(y). \tag{10}$$

Let us observe that since

$$P(y) - P(x) = (P(y) \oplus P(x)) \cap P(y),$$

hence, in view of (10), (6) and (1), for every two elements x and y of B,

$$P(y) - P(x) = P(y+xy). \tag{11}$$

Also, in view of (8), for every positive natural number n,

$$x = \sum_{i=1}^{n} c_i \quad \textit{implies} \quad P(x) \subset \bigcup_{i=1}^{n} P(c_i) \tag{12}$$

where c_i is an element of B. Moreover, in view of (1) and (7),

$$P(ca) \subset P(a), \tag{13}$$

where c and a are any two elements of B.

Now, let $\mathscr{P}$ represent *the set of all proper prime ideals of* B.

THEOREM 1. — *The Boolean ring B is isomorphic to a subring of the algebra of all subsets of $\mathscr{P}$.*

PROOF. — In view of (5), (6) and (10), the mapping f from B into the power set of $\mathscr{P}$, given by

$$f(x) = P(x)$$

establishes the desired isomorphism.

Next, in view of (6), we introduce a topology $\mathscr{T}$ in $\mathscr{P}$ such that, for every $x \in B$ the subset $P(x)$ of $\mathscr{P}$ is a basis element of $\mathscr{T}$.

DEFINITION 2. — *The topological space $(\mathscr{P}, \mathscr{T})$ is called the Stone space of B.*

LEMMA 2. — *In the space $(\mathscr{P}, \mathscr{T})$, every basis element is closed.*

PROOF. — Let $P(x)$ be a basis element and let $p(y) \notin P(x)$. Clearly, $p(y) \in (P(y) - P(x))$ and hence in view of (11),

$$p(y) \in P(y+xy).$$

Thus, an element $p(y)$, in the complement $\mathscr{P} - P(x)$ of $P(x)$, is contained in a basis element $P(y+xy)$ which is disjoint from $P(x)$. Hence $P(x)$ is closed.

LEMMA 3. — *The space $(\mathscr{P}, \mathscr{T})$ is totally disconnected.*

PROOF. — Let $p(x)$ and $p(y)$ be two distinct elements of $\mathscr{P}$. Thus, there exists $z \in B$ such that, say, $z \in p(x)$ and $z \notin p(y)$. But then $P(yz)$ is a basis element containing $p(y)$ and not containing $p(x)$. Consequently, in view of Lemma 2, every two distinct elements $p(x)$ and $p(y)$ of $\mathscr{P}$ are contained in two mutually disjoint closed sets of $(\mathscr{P}, \mathscr{T})$ whose union is $\mathscr{P}$. Thus, $(\mathscr{P}, \mathscr{T})$ is totally disconnected (and in particular, Hausdorff).

Lemma 4. — *The space $(\mathscr{P}, \mathscr{T})$ is locally compact.*

Proof. — It is sufficient to prove that every basis element $P(x)$ of $(\mathscr{P}, \mathscr{T})$ is compact. Now, let $A \subset B$ and $\bigcup_{y \varepsilon A} P(y)$ be a covering of $P(x)$, i.e.,

$$P(x) \subset \bigcup_{y \varepsilon A} P(y) \tag{14}$$

Let (A) denote the ideal generated by the elements of A. Claim that $x \varepsilon (A)$. Assume the contrary that $x \notin (A)$. But then, in view of Lemma 1, there exists a prime ideal $p(x)$ such that $(A) \subset p(x)$, and therefore, $p(x) \notin \bigcup_{y \varepsilon A} P(y)$, contradicting (14). Hence, our assumption is false and indeed, $x \varepsilon (A)$. Consequently, there exists a natural number n such that

$$x = \sum_{i=1}^{n} (m_i + b_i)\, a_i$$

where m_i is an integer, $b_i \varepsilon B$ and $a_i \varepsilon A$. But then, in view of (12) and (13),

$$P(x) \subset \bigcup_{i=1}^{n} P\big((m_i + b_i)\, a_i\big) \subset \bigcup_{y \varepsilon A} P(y)$$

asserting that, in view of (14), $\bigcup_{i=1}^{n} P\big((m_i + b_i)\, a_i\big)$ is a finite subcover of an arbitrary cover $\bigcup_{y \varepsilon A} P(y)$ of $P(x)$. Thus, indeed, $P(x)$ is compact and $(\mathscr{P}, \mathscr{T})$ is locally compact.

Finally, in view of Lemmas 3 and 4, and Theorem 1, we have,

Theorem 2. — *Every Boolean ring is isomorphic to a subring of the algebra of all subsets of its Stone space which is totally disconnected and locally compact.*

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