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# CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES

by Thomas E. MOTT

In a recent paper [1], R. L. Kruse and J. J. Deeley have proved an interesting theorem concerning the continuity of a real valued function of several real variables, when that function is continuous in each variable separately and satisfies a certain monotonicity condition. The proof given by Kruse and Deeley involves induction on the variables, however a somewhat shorter and simpler proof is given below. In addition, two interesting corollaries are stated.

**THEOREM 1.** — *Let  $f(x_1, \dots, x_n)$  be a real valued function defined on an open set  $G \subseteq R^n$ , and suppose that :*

- (i) *Whenever  $n - 1$  of the variables are fixed,  $f$  is a continuous function of the remaining variable.*
- (ii) *For each permissible <sup>1)</sup> value of  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  in  $R^{n-1}$  the function  $f(x_1, \dots, x_n)$  is a monotone function of  $x_i$ , the direction of monotonicity being dependent upon the choice of the point  $(x_1, \dots, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  in  $R^{n-1}$ ; all for  $i = 1, \dots, n$ .  
Then  $f(x_1, \dots, x_n)$  is continuous in  $G$ .*

*Proof:* Let  $(x_{1,0}, \dots, x_{n,0})$  be any point in  $G$ , then  $G$  being an open set we may choose  $\delta > 0$  such that the rectangle  $S = [x_{1,0} - \delta, x_{1,0} + \delta] \times \dots \times [x_{n,0} - \delta, x_{n,0} + \delta]$  is contained in  $G$ . In view of (i), given  $\epsilon > 0$  we may choose  $\delta_1$  in  $(0, \delta)$  such that

$$|f(x_1, x_{2,0}, \dots, x_{n,0}) - f(x_{1,0}, x_{2,0}, \dots, x_{n,0})| < \frac{\epsilon}{n}$$

whenever  $|x_1 - x_{1,0}| \leq \delta_1$ ,  $\delta_2$  in  $(0, \delta_2)$  such that

$$|f(x_{1,0} \pm \delta_1, x_2, x_3, \dots, x_{n,0}) - f(x_{1,0} \pm \delta_1, x_{2,0}, x_3, \dots, x_{n,0})| < \frac{\epsilon}{n}$$

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<sup>1)</sup> Permissible values of  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  in  $R^{n-1}$  being those for which  $(x_1, \dots, x_n) \in G$ .

whenever  $|x_2 - x_{2,0}| \leq \delta_2$ , and continuing in this manner we finally choose  $\delta_n$  in  $(0, \delta)$  such that

$$|f(x_{1,0} \pm \delta_1, x_{2,0} \pm \delta_2, \dots, x_{n-1,0} \pm \delta_{n-1}, x_n) - f(x_{1,0} \pm \delta_1, x_{2,0} \pm \delta_2, \dots, x_{n-1,0} \pm \delta_{n-1}, x_{n,0})| < \frac{\varepsilon}{n}$$

whenever  $|x_n - x_{n,0}| \leq \delta_n$ .

Let  $\bar{S} = [x_{1,0} - \delta_1, x_{1,0} + \delta_1] \times \dots \times [x_{n,0} - \delta_n, x_{n,0} + \delta_n]$ , then romf (ii) it follows that the function  $f$  assumes its maximum and minimum values at vertices of  $\bar{S}$ , let  $(x_{1,0} + \delta_1^*, \dots, x_{n,0} + \delta_n^*)$  and  $(x_{1,0} + \delta_1^{**}, \dots, x_{n,0} + \delta_n^{**})$  be these maximum and minimum points respectively, then  $\delta_i^* = \pm \delta_i$  and  $\delta_i^{**} = \delta_i$  for  $i = 1, \dots, n$  and certain choices of the  $\pm$  signs.

Now

$$\begin{aligned} & |f(x_{1,0} + \delta_1^*, \dots, x_{n,0} + \delta_n^*) - f(x_{1,0}, \dots, x_{n,0})| \leq \\ & |f(x_{1,0} + \delta_1^*, \dots, x_{n-1,0} + \delta_{n-1}^*, x_{n,0} + \delta_n^*) - \\ & \quad - f(x_{1,0} + \delta_1^*, \dots, x_{n-1,0} + \delta_{n-1}^*, x_{n,0})| + \\ & \quad + |f(x_{1,0} + \delta_1^*, \dots, x_{n-1,0} + \delta_{n-1}^*, x_{n,0}) - \\ & \quad - f(x_{1,0} + \delta_1^*, \dots, x_{n-2,0} + \delta_{n-2}^*, x_{n-1,0}, x_{n,0})| + \\ & \quad + \dots + |f(x_{1,0} + \delta_1^*, x_{2,0}, \dots, x_{n,0}) - f(x_{1,0}, \dots, x_{n,0})| < \varepsilon \end{aligned}$$

and similarly  $|f(x_{1,0} + \delta_1^{**}, \dots, x_{n,0} + \delta_n^{**}) - f(x_{1,0}, \dots, x_{n,0})| < \varepsilon$ . Therefore,  $|f(x_{1,0} + \delta_1^*, \dots, x_{n,0} + \delta_n^*) - f(x_{1,0} + \delta_1^{**}, \dots, x_{n,0} + \delta_n^{**})| < 2\varepsilon$  and consequently if  $(x'_1, \dots, x'_n)$ ,  $(x''_1, \dots, x''_n)$  are any two points of  $\bar{S}$  then  $|f(x'_1, \dots, x'_n) - f(x''_1, \dots, x''_n)| < 2\varepsilon$ . Since  $\varepsilon$  is arbitrary it now follows from the Cauchy Criterion that the function  $f$  is continuous at the point  $(x_{1,0}, \dots, x_{n,0})$  in  $G$ .

Two rather interesting results which follow directly from this theorem are:

*Corollary 1:* Let  $f(x_1, \dots, x_n)$  be a real valued function defined on an open set  $G \subseteq R^n$ . Let  $T$  be an invertible mapping from  $G$  into  $R^n$  defined by the equations  $u_i = p_i(x_1, \dots, x_n)$  ( $i=1, \dots, n$ ) in such a manner that the inverse mapping  $T^{-1}$  is defined by the equations  $x_i = q_i(u_1, \dots, u_n)$  ( $i=1, \dots, n$ ) where the functions  $p_i(x_1, \dots, x_n)$  ( $i=1, \dots, n$ ) are continuous in  $G$  and the functions  $q_i(u_1, \dots, u_n)$  ( $i=1, \dots, n$ ) are continuous in  $T(G)$ . Suppose that:

- (i) The function  $f$  is continuous along that portion of the curves  $\{ x_1 = q_1(u_1+t, u_2, \dots, u_n), \dots, x_n = q_n(u_1+t, u_2, \dots, u_n) \}, \dots, \{ x_1 = q_1(u_1, \dots, u_{n-1}, u_n+t), \dots, x_n = q_n(u_1, \dots, u_{n-1}, u_n+t) \}$  which lie in  $G$ , for every  $(u_1, \dots, u_n)$  in  $T(G)$ .
- (ii) For each permissible <sup>1)</sup> value of  $(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$  in  $R^{n-1}$  the function  $f(q_1(u_1, \dots, u_n), \dots, q_n(u_1, \dots, u_n))$  is a monotonic function of  $u_i$ , the direction of monotonicity being dependent upon the choice of the point  $(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$  in  $R^{n-1}$ ; all for  $i = 1, \dots, n$ . Then  $f(x_1, \dots, x_n)$  is continuous in  $G$ .

*Corollary 2:* Let  $f(x_1, \dots, x_n)$  be a real valued function defined on an open set  $G \subseteq R^n$  and let  $v_i = (\lambda_{i,1}, \dots, \lambda_{i,n})$  ( $i=1, \dots, n$ ) be linearly independent vectors in  $R^n$ . If the function  $f$  is continuous along that portion of every line passing through  $G$  and parallel to  $v_i$  ( $i=1, \dots, n$ ), and  $f$  is monotonic along each of these lines (the direction of monotonicity depending upon the choice of line), then  $f(x_1, \dots, x_n)$  is continuous in  $G$ .

#### REFERENCES

- [1] KRUSE, R. L. and J. J. DEELY, "Joint Continuity of Monotonic Functions, *Amer Math. Monthly*, 7» (1969), pg. (74-76).

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<sup>1)</sup> Permissible values of  $(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$  in  $R^{n-1}$  being those for which  $(u_1, \dots, u_n) \in T(G)$ .