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CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES

by Thomas E. MOTT

In a recent paper [1], R. L. Kruse and J. J. Deeley have proved an interesting theorem concerning the continuity of a real valued function of several real variables, when that function is continuous in each variable separately and satisfies a certain monotonicity condition. The proof given by Kruse and Deeley involves induction on the variables, however a somewhat shorter and simpler proof is given below. In addition, two interesting corollaries are stated.

THEOREM 1. — *Let $f(x_1, ..., x_n)$ be a real valued function defined on an open set $G \subseteq R^n$, and suppose that:*

(i) *Whenever $n-1$ of the variables are fixed, f is a continuous function of the remaining variable.*

(ii) *For each permissible* [1]) *value of $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$* in R^{n-1} *the function $f(x_1, ..., x_n)$ is a monotone function of x_i, the direction of monotonicity being dependent upon the choice of the point $(x_1, ..., ..., x_{i-1}, x_{i+1}, ..., x_n)$ in R^{n-1}; all for $i = 1, ..., n$.*
Then $f(x_1, ..., x_n)$ is continuous in G.

Proof: Let $(x_{1,0}, ..., x_{n,0})$ be any point in G, then G being an open set we may choose $\delta > 0$ such that the rectangle $S = [x_{1,0}-\delta, x_{1,0}+\delta] \times \times ... \times [x_{n,0}-\delta, x_{n,0}+\delta]$ is contained in G. In view of (i), given $\epsilon > 0$ we may choose δ_1 in $(0, \delta)$ such that

$$|f(x_1, x_{2,0}, \ldots, x_{n,0}) - f(x_{1,0}, x_{2,0}, \ldots, x_{n,0})| < \frac{\varepsilon}{\mathrm{n}}$$

whenever $|x_1 - x_{1,0}| \leqq \delta_1$, δ_2 in $(0, \delta_2)$ such that

$$|f(x_{1,0} \pm \delta_1, x_2, x_{3,0}, \ldots, x_{n,0}) - f(x_{1,0} \pm \delta_1\, x_{,2,0}, x_{3,0}, \ldots, x_{n,0})| < \frac{\varepsilon}{n}$$

1) Permissible values of $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ in R^{n-1} being those for which $(x_1, ..., x_n) \varepsilon G$.

whenever $|x_2 - x_{2,0}| \leqq \delta_2$, and continuing in this manner we finally choose δ_n in $(0, \delta)$ such that

$$|f(x_{1,0} \pm \delta_1, x_{2,0} \pm \delta_2, \dots, x_{n-1,0} \pm \delta_{n-1}, x_n) -$$

$$f(x_{1,0} \pm \delta_1, x_{2,0} \pm \delta_2, \dots, x_{n-1,0} \pm \delta_{n-1}, x_{n,0})| < \frac{\varepsilon}{n}$$

whenever $|x_n - x_{n,0}| \leqq \delta_n$.

Let $\bar{S} = [x_{1,0}-\delta_1, x_{1,0}+\delta_1] \times \dots \times [x_{n,0}-\delta_n, x_{n,0}+\delta_n]$, then romf (ii) it follows that the function f assumes its maximum and minimum values at vertices of $\bar{S}$, let $(x_{1,0}+\delta_1^*, \dots, x_{n,0}+\delta_n^*)$ and $(x_{1,0}+\delta_1^{**}, \dots, x_{n,0}+\delta_n^{**})$ be these maximum and minimum points respectively, then $\delta_i^* = \pm\delta_i$ and $\delta_i^{**} = \delta_i$ for $i = 1, \dots, n$ and certain choices of the $\mp$ signs.

Now

$$|f(x_{1,0} + \delta_1^*, \dots, x_{n,0} + \delta_n^*) - f(x_{1,0}, \dots, x_{n,0}) \leqq$$

$$|f(x_{1,0} + \delta_1^*, \dots, x_{n-1,0} + \delta_{n-1}^*, x_{n,0} + \delta_n^*) -$$

$$- f(x_{1,0} + \delta_1^*, \dots, x_{n-1,0} + \delta_{n-1}^*, x_{n,0})| +$$

$$+ |f(x_{1,0} + \delta_1^*, \dots, x_{n-1,0} + \delta_{n-1}^*, x_{n,0}) -$$

$$- f(x_{1,0} + \delta_1^*, \dots, x_{n-2,0} + \delta_{n-2}^*, x_{n-1,0}, x_{n,0})| +$$

$$+ \dots + |f(x_{1,0} + \delta_1^*, x_{2,0}, \dots, x_{n,0}) - f(x_{1,0}, \dots, x_{n,0})| < \varepsilon$$

and similarly $|f(x_{1,0}+\delta_1^{**}, \dots, x_{n,0}+\delta_n^{**}) - f(x_{1,0}, \dots, x_{n,0})| < \varepsilon$. Therefore, $|f(x_{1,0}+\delta_1^*, \dots, x_{n,0}+\delta_n^*) - f(x_{1,0}+\delta_1^{**}, \dots, x_{n,0}+\delta_n^{**})| < 2\varepsilon$ and consequently if $(x_1', \dots, x_n')$, $(x_1'', \dots, x_n'')$ are any two points of $\bar{S}$ then $|f(x_1', \dots, x_n') - f(x_1'', \dots, x_n'')| < 2\varepsilon$. Since $\in$ is arbitrary it now follows from the Cauchy Criterion that the function f is continuous at the point $(x_{1,0}, \dots, x_{n,0})$ in G.

Two rather interesting results which follow directly from this theorem are:

Corollary 1: Let $f(x_1, \dots, x_n)$ be a real valued function defined on an open set $G \subseteq R^n$. Let T be an invertible mapping from G into R^n defined by the equations $u_i = p_i(x_1, \dots, x_n)$ $(i=1, \dots, n)$ in such a manner that the inverse mapping T^{-1} is defined by the equations $x_i = q_i(u_1, \dots, u_n)$ $(i= =1, \dots, n)$ where the functions $p_i(x_1, \dots, x_n)$ $(i=1, \dots, n)$ are continuous in G and the functions $q_i(u_1, \dots, u_n)$ $(i=1, \dots, n)$ are continuous in $T(G)$. Suppose that:

(i) The function f is continuous along that portion of the curves $\{ x_1 = q_1 (u_1+t, u_2, ..., u_n), ..., x_n = q_n (u_1+t, u_2, ..., u_n) \}$, ..., $\{ x_1 = q_1 (u_1, ..., u_{n-1}, u_n+t), ..., x_n = q_n (u_1, ..., u_{n-1}, u_n+t) \}$ which lie in G, for every $(u_1, ..., u_n)$ in $T(G)$.

(ii) For each permissible [1]) value of $(u_1, ..., u_{i-1}, u_{i+1}, ..., u_n)$ in R^{n-1} the function $f(q_1 (u_1, ..., u_n), ..., q_n (u_1, ..., u_n))$ is a monotonic function of u_i, the direction of monotonicity being dependent upon the choice of the point $(u_1, ..., u_{i-1}, u_{i+1}, ..., u_n)$ in R^{n-1}; all for $i = 1, ..., n$. Then $f(x_1, ..., x_n)$ is continuous in G.

Corollary 2: Let $f(x_1, ..., x_n)$ be a real valued function defined on an open set $G \subseteq R^n$ and let $v_i = (\lambda_{i,1}, ..., \lambda_{i,n})$ $(i=1, ..., n)$ be linearly independent vectors in R^n. If the function f is continuous along that portion of every line passing through G and parallel to v_i $(i=1, ..., n)$, and f is monotonic along each of these lines (the direction of monotonicity depending upon the choice of line), then $f(x_1, ..., x_n)$ is continuous in G.

REFERENCES

[1] KRUSE, R. L. and J. J. DEELY, " Joint Continuity of Monotonic Functions, *Amer Math. Monthly*, 7» (1969), pg. (74-76).

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[1]) Permissible values of $(u_1, ..., u_{i-1}, u_{i+l}, ..., u_n)$ in R^{n-1} being those for which $(u_1, ..., u_n) \varepsilon T(G)$.