A NOTE ON ABSOLUTE SUMMABILITY FACTORS

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In this paper results which appear in [2] are extended, namely Theorem 3. The notation used here and explanation of the problems are given there. The notation $\varepsilon_\nu \in (| A |, | B |)_r$ means $\sum a_\nu \varepsilon_\nu \in | B |$ whenever $\sum a_\nu \in | A |$.

The following theorem is a special case of Theorem 3 in [2]. However, strictly speaking somewhat stronger, since it does not require the representation (4) in [2]. This is important later on.

THEOREM 1. If A is absolutely regular, normal, $A' \overset{\cdot}{\leqq} 0$ and $a_{\nu\nu} > 0\downarrow$ for $\nu\uparrow$ ($A' \overset{\cdot}{\leqq} 0$ and $a_{nn} > 0$ imply $A \overset{\cdot}{\geqq} 0$, see footnote 1) in [2]), then $\varepsilon_\nu = 0\,(a_{\nu\nu})$ implies $\varepsilon_\nu \in (| A |, | I |)_r$.

Proof. It suffices to prove $\sum_{n=\nu}^{\infty} | a'_{n\nu}\varepsilon_n | \leqq M$. Since $\sum_{n=\nu}^{\infty} | a'_{n\nu} | = \frac{2}{a_{\nu\nu}} - 1$ (see, Lemma 1 in [1]), and $| \varepsilon_n | \leqq K a_{nn}$, we have $\sum_{n=\nu}^{\infty} | a'_{n\nu}\,\varepsilon_n | \leqq 2K$ using $a_{nn}\downarrow$.

Proofs of the following lemmas can be found in [1]. The notation $| B | \subseteq | A |$ means $\sum a_\nu \in | A |$ whenever $\sum a_\nu \in | B |$.

Lemma 1. If A and B are absolutely regular and normal, and if $| B | \subseteq \subseteq | A |$, then given a bounded sequence $c = \{ c_\nu \}$ there exists a bounded sequence $c' = \{ c'_\nu \}$ such that $\varepsilon_\nu (A, c) = \varepsilon_\nu (B, c')$.

Lemma 2. Let $A = BP$, P a weighted mean. Then,

$$[\varepsilon_{\nu+1} (A, c) - \varepsilon_\nu (A, c)] \frac{P_{\nu-1}}{p_\nu} = \varepsilon_\nu (A, c) - \varepsilon_\nu (B, c) .$$

We now extend Theorem 1 to matrices of the form AP^k where k is a positive integer and P is a weighted mean. If P is a weighted mean, then $p_{n\nu} = \frac{p_n}{P_n P_{n-1}} P_{\nu-1}$ $(p_{00}=1)$.

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THEOREM 2. Suppose P is a weighted mean with $p_\nu > 0$, $a_{\nu\nu} > 0\downarrow$ $(\nu\uparrow)$, and $p_{\nu+1} = 0(p_\nu)$. If $\varepsilon_\nu = 0(a_{\nu\nu})$ implies $\varepsilon_\nu \in (|A|, |I|)_r$, then $\delta_\nu = 0\left(\frac{a_{\nu\nu}\, p_\nu}{P_\nu}\right)$ implies $\delta_\nu \in (|AP|, |I|)_r$.

Proof. It suffices to prove $\sum |a_\nu\, a_{\nu\nu} \frac{p_\nu}{P_\nu}| < \infty$ whenever $\sum a_\nu \in |AP|$. Using the identity

$$a_\nu = \frac{1}{P_{\nu-1}} \sum_{k=1}^{\nu} P_{k-1} a_k - \frac{1}{P_{\nu-1}} \sum_{k=1}^{\nu-1} P_{k-1} a_k \, (\nu \geqq 1)$$

we conclude

$$\sum_{\nu=1}^{\infty} | a_\nu \frac{a_{\nu\nu}\, p_\nu}{P_\nu} | \leqq \sum_{\nu=1}^{\infty} | a_{\nu\nu} P_\nu(a) | + \sum_{\nu=1}^{\infty} | \frac{a_{\nu+1,\,\nu+1}}{a_{\nu\nu}} \cdot \frac{p_{\nu+1}}{p_\nu} \cdot a_{\nu\nu} P_\nu(a) |$$

($P_\nu(a)$ denotes the P-transform of $\sum a_\nu$.) Since, $\sum P_\nu(a) \in |A|$ we have $\sum |a_{\nu\nu} P_\nu(a)| < \infty$ by hypothesis. The same holds true for the second sum, since all of the additional factors are bounded. This completes the proof. To conclude the results of Theorem 2 with AP^k (k an integer, $k > 1$) in place of AP we need only show $a_{\nu+1,\nu+1} = 0(a_{\nu\nu})$ implies $(AP^k)_{\nu+1,\nu+1} = 0((AP^k)_{\nu\nu})$. This is done inductively using $p_{\nu+1} = 0(p_\nu)$. Thus we have by induction.

THEOREM 3. Suppose P is a weighted mean which satisfies the hypotheses of Theorem 2. If $\varepsilon_\nu = 0(a_{\nu\nu})$ implies $\varepsilon_\nu \in (|A|, |I|)_r$, then $\delta_\nu = = 0\left(a_{\nu\nu}\left(\frac{p_\nu}{P_\nu}\right)^k\right)$ implies $\delta_\nu \in (|AP^k, |I|)_r$.

We now state and prove the main result of the paper.

THEOREM 4. Suppose A and B are normal, and absolutely regular. Let P be a weighted mean with $p_\nu > 0$ and $|AP| \supseteq |A|$. If

(1) $$\delta_\nu = 0(a_{\nu\nu}) \textit{ implies } \delta_\nu \in (|A|, |I|)_r$$

and

(2) $$\varepsilon_\nu(A, c) = 0\left(\frac{a_{\nu\nu}}{b_{\nu\nu}}\right) \textit{ implies } \varepsilon_\nu \in (|A|, |B|)_r,$$

then

$$\varepsilon_\nu(AP, c) = 0\left(\frac{a_{\nu\nu}\, p_\nu}{b_{\nu\nu}\, P_\nu}\right) \textit{ implies } \varepsilon_\nu(AP, c) \in (|AP|, |B|)_r$$

whenever

$$b_{n\nu} \uparrow \textit{ for } \nu \uparrow (\nu \leqq n),$$

$$a_{\nu+1,\,\nu+1} = 0(a_{\nu\nu}),\ b_{\nu\nu} = 0(b_{\nu+1,\,\nu+1}),\ \textit{and}\ p_{\nu+1} = 0(p_\nu).$$

Proof. If $\sum a_\nu \in | AP |$, then $\sum P_\nu(a) \in | A |$. Since $p_\nu > 0$, we have $a_\nu = \sum_{\mu=0}^{\nu} p'_{\nu\mu} \alpha_\mu$ where $\sum \alpha_\mu \in | A |$. Using this we have

$$\beta_n = \sum_{\nu=0}^{n} b_{n\nu}\, \varepsilon_\nu(AP, c)\, a_\nu = \sum_{\mu=0}^{n} \alpha_\mu \sum_{\nu=\mu}^{n} b_{n\nu}\, \varepsilon_\nu(AP, c)\, p'_{\nu\mu}.$$

The matrix $P' = (p'_{\nu\mu})$ is given by

$$p'_{\nu\nu} = \frac{P_\nu}{p_\nu},\ p'_{\nu,\nu-1} = -\frac{P_{\nu-2}}{p_{\nu-1}},\ p_{\nu\mu} = 0 \textit{ otherwise.}$$

Introducing the inverse we have

$$\beta_n = \sum_{\mu=0}^{n} \alpha_\mu\, b_{n\mu}\, \varepsilon_\mu(AP, c) + \sum_{\mu=0}^{n} \alpha_\mu \frac{P_{\mu-1}}{p_\mu} (b_{n\mu}\, \varepsilon_\mu(AP, c) -$$

$$- b_{n\mu+1} \varepsilon_{\mu+1}(AP, c)).$$

Write

$$b_{n\mu}\, \varepsilon_\mu(AP, c) - b_{n,\mu+1}\, \varepsilon_{\mu+1}(AP, c) = b_{n\mu}(\varepsilon_\mu(AP, c) - \varepsilon_{\mu+1}(AP, c)) +$$

$$\varepsilon_{\mu+1}(AP, c)(b_{n\mu} - b_{n\mu+1}).$$

From this it follows by Lemmas 1 and 2 that

$$(3) \qquad \sum_{\nu=0}^{n} b_{n\nu}\, \varepsilon_\nu(AP, c)\, a_\nu = \sum_{\mu=0}^{n} b_{n\mu}\, \varepsilon_\mu(A, c')\, \alpha_\mu +$$

$$\sum_{\mu=0}^{n} \alpha_\mu \frac{P_{\mu-1}}{p_\mu}\, \varepsilon_{\mu+1}(AP, c)(b_{n\mu} - b_{n\mu+1}),$$

where $\varepsilon_\nu(A, c') = 0\left(\frac{a_{\nu\nu}}{b_{\nu\nu}}\right)$ by Lemma 2. Thus by (2) the first term is absolutely convergent. Write the second term of (3) in the form $\sum_{\mu=0}^{n} a_{\mu\mu}\, \alpha_\mu\, A_{n\mu}$, where

$$A_{n\mu} = \frac{1}{a_{\mu\mu}} \frac{P_{\mu-1}}{p_\mu} \times \varepsilon_{\mu+1}(AP, c)(b_{n\mu} - b_{n\mu+1}).$$

$$\sum_{n=\mu}^{\infty} | A_{n\mu} | = 0\left(\frac{b_{\mu\mu}}{a_{\mu\mu}} \frac{P_{\mu-1}}{P_\mu} \frac{a_{\mu+1,\mu+1}}{b_{\mu+1,\mu+1}} \frac{p_{\mu+1}}{p_\mu}\right)$$

By hypothesis (1) $\sum | a_{\mu\mu} \alpha_{\mu} | < \infty$. Hence the second term of (3) is absolutely convergent if $\sum_{n=\mu}^{\infty} | A_{n\mu} | \leqq M$. Since $b_{n\mu}\uparrow$ for $\mu\uparrow$ we have

which is 0 (1).

THEOREM 5. Let A, B be normal and absolutely regular with $A \dot{>} 0$, $B \dot{>} 0$, and $A' \dot{\leqq} 0$. Furthermore, assume

$$a_{n\nu}\uparrow\,, \quad \nu\uparrow \ (\nu \leqq n) \tag{4}$$

$$b_{n\nu}\uparrow\,, \quad \nu\uparrow \ (\nu \leqq n) \tag{5}$$

$$\frac{a_{n\nu}}{b_{n\nu}} \downarrow, \quad \nu\uparrow \ (\nu \leqq n) \tag{6}$$

$$\frac{b_{n\nu}}{a_{n\nu}} (a_{kn} - a_{k\nu}) \downarrow\,, \quad \nu\uparrow \textit{ for all } n \leqq k \tag{7}$$

$$a_{\nu+1,\nu+1} = 0 (a_{\nu\nu}) \tag{8}$$

$$b_{\nu\nu} = 0 (b_{\nu+1,\nu+1}) \tag{9}$$

$$p_{\nu+1} = 0 (p_{\nu})\,. \tag{10}$$

When these conditions are satisfied

$$\varepsilon_{\nu} (AP^{k}, c) = 0 \left(\frac{a_{\nu\nu}}{b_{\nu\nu}} \left(\frac{p_{\nu}}{P_{\nu}} \right)^{k} \right)$$

implies

$$\varepsilon_{\nu} (AP^{k}, c) \in (| AP^{k} |, | B |)_{r}\,.$$

Proof. With $k = 0$ these condition imply $\varepsilon_{\nu} (A,c) \in (|A|, |B|)_{r}$ (see Theorem 3 in [2]. Hence, the theorem follows by induction from Theorem 3 and 4.

Theorem 5 extends Theorem 3 in [2] to include all Cesaro methods $A = (C, \alpha)$, $B = (C, \beta)$ where $\alpha \geqq 0$, and $0 \leqslant \beta \leqslant 1$.

BIBLIOGRAPHY

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