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ly, let $\mathscr{A}$ be a sheaf of $\mathcal{O}_Y$-algebras, which is coherent as sheaf of $\mathcal{O}_Y$-modules. Then there exists an analytic space $(X, \mathcal{O}_X)$ and a finite morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f_* (\mathcal{O}_X)$ is isomorphic with $\mathscr{A}$ as sheaf of $\mathcal{O}_Y$-algebras ; the triple $(X, \mathcal{O}_X, f)$ is unique up to an isomorphism.

We do not prove this proposition here and refer to Houzel [6] or Narasimhan [9] for this proof. We note also that a proof of the direct part can be given along the same lines as theorem 3.1.3, combined with the fact that direct images under finite morphism preserve exact sequences of sheaves of $\mathcal{O}_X$-modules (in other words, that higher direct images are zero). We note also that, for *proper* morphisms (not necessarily finite), a much deeper result has been proved by Grauert [2], [3].

Finally, we remark that, in the real case, proposition 3.3.2. is false (take, for instance, X the submanifold of $\mathbf{R}^2$ defined by $x_2 - x_1^2 = 0$, $Y = \mathbf{R}$ and $f =$ the projection on the x_2-axis ; $f_* (\mathcal{O}_X)$ has support $x_2 \geqslant 0$, which is not an analytic subset of $\mathbf{R}$, hence $f_* (\mathcal{O}_X)$ cannot be coherent!)

CHAPTER 4.

THE FINITENESS THEOREM

In this chapter, we consider only *complex* analytic spaces, separated and having a countable basis of open sets.

4.1. *Stein spaces*

Let $(X, \mathcal{O}_X)$ be an analytic space, and K a subset of X ; we denote, as usual by $\hat{K}$ the set

$$\left\{ x \in X \mid \forall f \in \Gamma(X, \mathcal{O}_x) : |f(x)| \leqslant \sup_{y \in K} |f(y)| \right\}$$

Definition 4.1.1. a) $(X, \mathcal{O}_X)$ is called holomorphically convex if, for any K compact $\subset X$, $\hat{K}$ is compact ;
b) $(X, \mathcal{O}_X)$ is called a Stein space if it is holomorphically convex, and if, for any $x \in X$, there exist sections $f_1, \ldots, f_p \in \Gamma(X, \mathcal{O}_X)$ with $f_i(x) = 0$, such that x is an isolated point of the counter-image of 0 in the morphism $(X, \mathcal{O}_X) \to \mathbf{C}^p$ defined by $f_1, \ldots, f_p$, (This last property can also by expressed as the fact that the morphism of germs : $(X, \mathcal{O}_X, x) \to (\mathbf{C}^p, 0)$ defined by $f_1, \ldots, f_p$ is finite).

If X is a Stein space, X_{red} is obviously also a Stein space. The converse is also true (see Grauert [2]).

Theorem 4.1.2. (" Theorems A and B " of Cartan-Oka). Let F be an analytic coherent sheaf over a Stein space $(X, \mathcal{O}_X)$. Then
1) For any $x \in X$, $\Gamma(X, F)$ generates F_x over $\mathcal{O}_{X,x}$
2) For $p \geqslant 1$, one has $H^p(X, F) = 0$

This theorem will not be proved here (see f.i. [5] for the reduced case ; the general case is similar). We will need here only the following special case :

Let $(X, \mathcal{O}_X)$ be a closed analytic subspace of a domain of holomorphy $U \subset \mathbf{C}^n$; if F is an analytic coherent sheaf on X, let $\tilde{F}$ be the trivial extension of F to U ; then $\tilde{F}$ is a coherent sheaf of $\mathcal{O}_U$ modules, and theorems A and B are valid for $\tilde{F}$: therefore, they are true for F.

4.2. *Topology on* $\Gamma(X, F)$.

1. Let X be a closed analytic subspace of a domain of holomorphy $U \subset \mathbf{C}^n$; and, with the previous notations, suppose that $\tilde{F}$ admits a *finite presentation* i.e. an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^q \xrightarrow{\alpha} \mathcal{O}_U^p \xrightarrow{\beta} \tilde{F} \to 0 .$$

Applying theorem B to the exact sequences

$$0 \to \operatorname{Im} \alpha \to \mathcal{O}_U^p \to \tilde{F} \to 0 \quad \text{and} \quad 0 \to \operatorname{Ker} \alpha \to \mathcal{O}_U^q \to \operatorname{Im} \alpha \to 0$$

we get an exact sequence

$$\Gamma(U, \mathcal{O}_U)^q \xrightarrow{\Gamma(U,\alpha)} \Gamma(U, \mathcal{O}_U)^p \xrightarrow{\Gamma(U,\beta)} \Gamma(U, \tilde{F}) \to 0 .$$

The space $\Gamma(U, \mathcal{O}_U)$, with the topology of uniform convergence on compact sets is a Frechet space. And we claim that, for that topology, $\operatorname{Im} \Gamma(U, \alpha)$ is closed. For, if f is adherent to $\operatorname{Im} \Gamma(U, \alpha)$, it results easily from Krull's theorem (see Appendix) that, for $x \in U$, we have $f_x \in \operatorname{Im}(\alpha_x)$, hence $f \in \Gamma(U, \operatorname{Im} \alpha)$; but, according to theorem B, the mapping $\Gamma(U, \mathcal{O}_U)^q \to \Gamma(U, \operatorname{Im} \alpha)$ is surjective.

Now, with the quotient topology, $\Gamma(X, F) \simeq \Gamma(U, \tilde{F}) \simeq \Gamma(U, \mathcal{O}_U) / \operatorname{Im} \Gamma(U, \alpha)$ is a Frechet space. This topology does not depend on the given presentation of $\tilde{F}$ (in fact, it does not even depend on the imbedding $X \to U$, but we shall not need it here). For, suppose we have a second presentation

$$\Gamma(U,\mathcal{O}_U)^{q'} \xrightarrow{\alpha'} \Gamma(U,\mathcal{O}_U)^{p'} \xrightarrow{\beta'} \tilde{F} \to 0\,.$$

As $\Gamma(U, \mathcal{O}_U)^p$ is free over $\Gamma(U, \mathcal{O}_U)$, we can find a $\Gamma(U, \mathcal{O}_U)$-linear map $\Gamma(U, \mathcal{O}_U)^p \xrightarrow{\gamma} \Gamma(U, \mathcal{O}_U)^{p'}$ such that $\beta = \beta' \circ \gamma$; this induces a continuous map

$$\Gamma(U,\mathcal{O}_U)^p/\mathrm{Im}\,\Gamma(U,\alpha) \to \Gamma(U,\mathcal{O}_U)^{p'}/\mathrm{Im}\,\Gamma(U,\alpha')$$

which is bijective, hence bicontinuous according to the closed graph theorem.

2. *General case*

If X is an analytic space and F an analytic coherent sheaf on X, we can find a) a locally finite covering of X by open subspaces X_i, b) for each i, a morphism $X_i \to U_i$, U_i open polycylinder in $\mathbf{C}^{n_i}$, which identifies X_i with a closed subspace of U_i c) for each i, a coherent sheaf $\tilde{F}_i$ on U_i admitting a finite presentation, such that $\tilde{F}_i$ is the extension of $F/_{X_i}$.

On $\Gamma(X_i, F|_{X_i})$ we have already defined a topology ; further, consider the natural injection

$$\Gamma(X,F) \to \prod_i \Gamma(X_i, F|_{X_i}))$$

We claim that its image is closed. For, (f_i) belongs to the image if and only if, for all $x \in X_i \cap X_j$ ($= X_i \times_X X_j$), we have $(f_i)_x = (f_j)_x$; and the fact that this relations define a closed subspace results easily from Krull's theorem.

This gives a topology of Frechet space on $\Gamma(X, F)$. It does not depend on the chosen covering (if one has tavo coverings, one considers a common refinement, and one applies again Krull's theorem and the closed graph theorem; we leave the details to the reader). One proves in the same way that if X' is an open subspace of X, the restriction map $\Gamma(X, F) \to \Gamma(X', F|_{X'})$ is continuous. If X' is relatively compact in X, then the restriction map is compact (this can be seen by choosing a covering X'_j of X' of the same type, such that, for any j, there exist i with $X'_j \subset X_i$, X'_j relatively compact in X_i, and applying Ascoli's theorem).

4.3. *Topology on $H^p(X, F)$*

We consider a locally finite covering $\mathcal{U} = \{X_i\}_{i \in I}$ by open subspaces of the preceding type. If we have $i_0, \ldots, i_p \in$ I, we consider the natural morphisms

$$X_{i_0 \ldots i_p} = X_{i_0} \underset{X}{\times} \cdots \underset{X}{\times} X_{i_p} \to X_{i_0} \times \cdots \times X_{i_p} \to U_{i_0} \times \cdots \times U_{i_p}$$

which makes $X_{i_0}, \ldots, {}_{i_p}$ isomorphic with a closed subspace of $U_{i_0} \times \ldots \times U_{i_p}$ (the hypothesis that X is separated is essential here! See remark at the end of this paragraph), therefore, $X_{i_0}, \ldots, {}_{i_p}$ satisfies theorems A and B ; more generally, if a finite number of open subspaces of X is Stein, their intersection is also Stein.

Introduce a total order on I. Given an analytic coherent sheaf on X, we can identify the alternating cochains of degree p of the covering $\mathcal{U}$ with values in F with the space

$$C^p(\mathcal{U},F) = \prod_{i_0 < i_1 < \ldots < i_p} \Gamma(X_{i_0 \ldots i_p}, F|x_{i_0 \ldots i_p}) .$$

This is a Frechet space, and the differential d: $C^p(\mathcal{U}, F) \to C^{p+1}(\mathcal{U}, F)$ is clearly continuous. Therefore the kernel $Z^p(\mathcal{U}, F)$ is a closed subspace of $C^p(\mathcal{U}, F)$. We denote $B^p(\mathcal{U},F)$ the image of $C^{p-1}(\mathcal{U},F)$ under d, and we consider on $H^p(\mathcal{U},F) = Z^p(\mathcal{U},F)/B^p(\mathcal{U}, F)$ the quotient topology; according to Leray's theorem, there is a natural isomorphism $H^p(X,F) \simeq H^p(\mathcal{U},F)$.

This gives a topology on $H^p(X, F)$ of a quotient of a Frechet space. In general, this topology is *not separated.*

We prove now that this topology is independent of the covering $\mathcal{U}$; to do that, it is sufficient to consider a refinement $\mathcal{U}' = \{ X'_j \}_{j\in J}$ of $\mathcal{U}$ of the same type, a map $\varphi : J \to I$ such that $X'_j \subset X_{\varphi(j)}$ for any j to consider the map defined by $\varphi : C^*(\mathcal{U},F) = \oplus_p C^p(\mathcal{U},F) \xrightarrow{p} C^*(\mathcal{U}',F)$ and to prove that the induced map $\bar{\rho} : H^p(\mathcal{U}, F) \to H^p(\mathcal{U}', F)$ is an isomorphism.

First, $\bar{\rho}$ is obviously continuous and bijective ; so, according to the closed graph theorem, all that we have to prove is that $\bar{\rho}$ maps the adherence of 0 onto the adherence of zero ; to do that, we consider $\bar{a}' \in H^p(\mathcal{U}, F)$, which is adherent to zero ; this means that $\bar{a}'$ is the class modulo $B^p(\mathcal{U}',F)$ of some $a' \in Z^p(\mathcal{U}' F)$ which is adherent to $B^p(\mathcal{U}', F)$; therefore, we have

$$a' = \lim_{n\to\infty} db'_n , \qquad b'_n \in C^{p-1}(\mathcal{U}', F) .$$

Now, the map

$$Z^p(\mathcal{U}, F) \oplus C^{p-1}(\mathcal{U}', F) \overset{(\rho,d)}{\to} Z^p(\mathcal{U}', F)$$

is surjective hence, according to the closed graph theorem, we can find converging sequences $a_n \in Z^p(\mathcal{U}, F)$ and $b''_n \in C^{p-1}(\mathcal{U}', F)$ such that $d\,b'_n = \rho(a_n) + d\,b''_n$; but, $\bar{\rho}$ being an isomorphism, we have $a_n = d\,\alpha_n$, $\alpha_n \in C^{n-1}(\mathcal{U}, F)$; if we put $b = \lim_{n\to\infty} b_n$, $a = \lim a_n$, we find that $a \in \overline{B^p(\mathcal{U}, F)}$ and that the class a of a is $H^p(\mathcal{U}, F)$ verifies $\bar{\rho}(\bar{a}) = \bar{a}'$; this proves the result.

Remark. If X is not separated, an intersection of two open Stein subspaces of X need not be Stein; take f.i. for X two copies of $\boldsymbol{C}^2$, identified everywhere except at O; there is an obvious covering of X by two open subspaces, identicals with $\boldsymbol{C}^2$; but their intersection is $\boldsymbol{C}^2 - \{o\}$, and therefore is not Stein!

4.4. *The finiteness theorem*

Theorem 4.4.1. (Cartan — Serre). Let X be a compact analytic space, and F be a coherent analytic sheaf on X. Then, for every $p \geqslant 0$ $H^p(X, F)$ is separated and finite dimensional.

We shall give two proofs of this theorem ; both are interesting for further applications.

1st proof. Let $\{X_i\}$ and $\{X_i'\}$ be two finite coverings of X of the type considered in the previous articles, such that, for every i, X'_i is relatively compact in X_i. Then, if we denote by $\mathcal{U}$ (resp. $\mathcal{U}'$) the covering $\{X_i\}$ (resp. $\{X_i'\}$), the natural restriction map $C^p(\mathcal{U}, F) \to C^p(\mathcal{U}', F)$ is compact.

Consider now the map

$$(\rho, d) : Z^p(\mathcal{U}, F) \oplus C^{p-1}(\mathcal{U}', F) \to Z^p(\mathcal{U}', F)$$

this map is surjective, and we have $(, d\rho) = (\rho, 0) + (0, d)$, $(\rho, 0)$ being compact ; then the following lemma proves that Im $(0, d)$ is closed and finite codimensional, q.e.d.

Lemma 4.4.2. Let E and F two Frechet spaces, u_1 and u_2 two linear continuous maps $E \to F$ such that $u_1 + u_2$ is surjective, and u_1 compact. Then Im (u_2) is closed and finite codimensional. For the proof, see e.g. [5].

2nd proof. Consider $\mathcal{U}$ and $\mathcal{U}'$ as above, and consider the map
(ρ, d) $C^{p-1}(\mathcal{U}, F)/Z^{p-1}(\mathcal{U}, F) \to [C^{p-1}(\mathcal{U}', F)/Z^{p-1}(\mathcal{U}', F)] \oplus Z^p(\mathcal{U}, F)$
(ρ, d) is clearly injective. I claim that its image is closed: In fact, since $\bar{\rho} : H^p(\mathcal{U}, F) \to H^p(\mathcal{U}', F)$ is injective, this image consists of the pairs $(\bar{a}', b)$, $a' \in C^{p-1}(\mathcal{U}', F)$, $b \in Z^p(\mathcal{U}, F)$ such that $da' = \rho\, b$, which proves the assertion.

Now we have $(\rho, d) = (\rho, 0) + (0, d)$ and $(\rho, 0)$ is compact. By a well-known lemma, it results that Im $(0, d)$ is closed, which means that $H^p(\mathcal{U}, F)$ is separated.

Finally, since $\bar{\rho}$ is compact, and is an isomorphism, it follows that the identity map of $H^p(\mathcal{U}, F)$ into itself is compact ; therefore this space is finite dimensional ; this proves the theorem.